Localization and universality phenomena for random polymers

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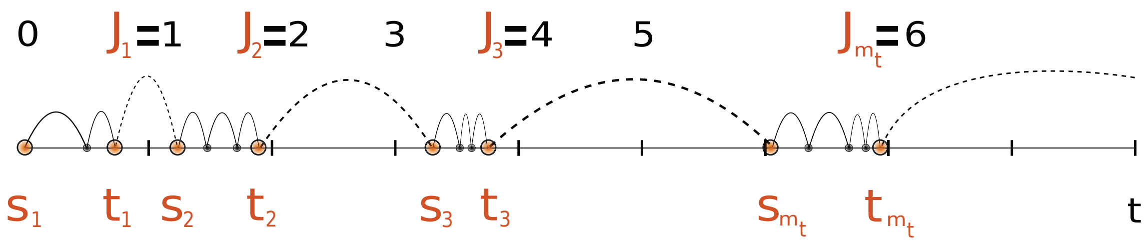
Phénomènes de localisation et d'universalité pour des polymères aléatoires



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Abstract

A polymer is a long chain of repeated units (monomers) that are almost identical, but they can differ in their degree of affinity for certain solvents. Such property allows to have interactions between the polymer and the external environment. The environment has only a region that can interact with the polymer. This interaction can attract or repel the polymer, by changing its spatial configuration, giving rise to localization and concentration phenomena. It is then possible to observe the existence of a phase transition. Whenever such region is a point or a line (but also a plane or hyper-plane), then we talk about pinning model, which represents the main subject of this thesis.

From a mathematical point of view, the pinning model describes the behavior of a Markov chain in interaction with a distinguished state. This interaction can attract or repel the Markov chain path with a force tuned by two parameters, h and β. If $\beta = 0$ we obtain the homogeneous pinning model, which is completely solvable. The disordered pinning model, which corresponds to $\beta > 0$, is most challenging and mathematically interesting. In this case the interaction depends on an external source of randomness, independent of the Markov chain, called disorder. The interaction is realized by perturbing the original Markov chain law via a Gibbs measure (which depends on the disorder, h and β), biasing the probability of a given path. Our main aim is to understand the structure of a typical Markov chain path under this new probability measure.

Pinning model with heavy-tailed disorder: The first research topic of this thesis is the pinning model in which the disorder is heavy-tailed and the return times of the Markov chain have a sub-exponential distribution. We prove that the set of the times at which the Markov chain visits the distinguished state, suitably rescaled, converges in distribution to a limit random set which depends only on the disorder. We show that there exists a phase transition with a random critical point, $\beta_c(h)$, below which the limit set is trivial. This work has interesting connections with the directed polymer in random environment with heavy tail.

Critical behavior of the pinning model in the weak coupling regime: We consider a pinning model with a light-tailed disorder and the return times of the Markov chain with a polynomial tail distribution, with exponent tuned by $\alpha > 0$. It is possible to show that there exists a non-trivial interaction between the parameters h and β. Such interaction gives rise to a critical point, $h_c(\beta)$, depending only on the law of the disorder and of the Markov chain. If $h > h_c(\beta)$, then the Markov chain visits infinitely many times the distinguished state and we say that it is localized. Otherwise, if $h < h_c(\beta)$, then the Markov chain visits such state only a finite number of times. Therefore the critical behavior of the model is deeply connected with the structure of $h_c(\beta)$. A very challenging problem is to describe the behavior of the pinning model in the weak disorder regime. To be more precise, one wants to understand the behavior of the critical point, $h_c(\beta)$, when $\beta \to 0$. The answer depends on the value of α: if it is smaller than $1/2$, then there exists a value of $\beta > 0$ below which the critical behavior of the model is the same of the homogeneous one. In this case we say that disorder is irrelevant. Otherwise, if $\alpha > 1/2$, whatever be the value of $\beta > 0$, the disorder perturbs the Markov chain, and we say that disorder is relevant. In the case of $1/2 < \alpha < 1$, in the literature there are non-matching estimates about the asymptotics of $h_c(\beta)$ as $\beta \to 0$. Getting the exact asymptotics for $h_c(\beta)$ represents the most important result of this thesis. We show that the behavior of the pinning model in the weak disorder limit is universal and the critical point, suitably rescaled, converges to the related quantity of a continuum model. The proof is obtained by using a coarse-graining procedure, which generalizes the technique developed for the copolymer model, a relative polymer model of the pinning one.

keywords: Pinning Model; Random Polymer; Directed Polymers; Weak Disorder; Scaling Limit; Disorder Relevance; Localization; Heavy Tails; Universality; Free Energy; Critical Point; Coarse-Graining

Résumé

D'un point de vue chimique et physique, un polymère est une chaîne d'unités répétées, appelées monomères, qui sont presque identiques, et chacune peut avoir un degré différent d'affinité avec certains solvants. Cette caractéristique permet d'avoir des interactions entre le polymère et le milieu dans lequel le polymère se trouve. Dans le milieu il y a une région interagissant, de manière attractive ou répulsive, avec le polymère. Cette interaction peut avoir un effet substantiel sur la structure du polymère, en donnant lieu à des phénomènes de localisation et de concentration. Il est donc possible observer l'existence d'une transition de phase. Quand cette région est un point ou une ligne (ou alors un plan ou un hyper-plan) on parle du modèle d'accrochage de polymère – pinning model – qui représente l'objet d'étude principal de cette thèse.

Mathématiquement le modèle d'accrochage de polymère décrit le comportement d'une chaîne de Markov en interaction avec un état donné. Cette interaction peut attirer ou repousser le chemin de la chaîne de Markov avec une force modulée par deux paramètres, h et β. Quand $\beta = 0$ on parle de modèle homogène, qui est complètement solvable. Le modèle désordonné, qui correspond à $\beta > 0$, est mathématiquement le plus intéressant. Dans ce cas l'interaction dépend d'une source d'aléa extérieur indépendant de la chaîne de Markov, appelée désordre. L'interaction est réalisée en modifiant la loi originelle de la chaîne de Markov par une mesure de Gibbs (dépendant du désordre, de h et de β), en changeant la probabilité d'une trajectoire donnée. La nouvelle probabilité obtenue définit le modèle d'accrochage de polymère. Le but principal est d'étudier et de comprendre la structure des trajectoires typiques de la chaîne de Markov sous cette nouvelle probabilité.

Modèle d'accrochage de polymère avec désordre à queues lourdes: Le premier sujet de recherche de cette thèse concerne le modèle d'accrochage de polymère où le désordre est à queues lourdes et où le temps de retour de la chaîne de Markov suit une distribution sous-exponentielle. Nous démontrons que l'ensemble des temps dans lesquels la chaîne de Markov visite l'état donné, avec un opportun changement d'échelle, converge en loi vers un ensemble limite qui dépend seulement du désordre. Nous démontrons qu'il existe une transition de phase avec un point critique aléatoire, $\beta_c(h)$, en dessous duquel l'ensemble limite est trivial. Ce travail a des connections intéressantes avec un autre modèle de polymère très répandu: le modèle de polymère dirigé en milieu aléatoire avec queues lourdes.

Comportement critique du modèle d'accrochage de polymère dans la limite du désordre faible: Nous étudions le modèle d'accrochage de polymère avec un désordre à queues légères et le temps de retour de la chaîne de Markov avec une distribution à queues polynomiales avec exposant caractérisé par $\alpha > 0$. Sous ces hypothèses on peut démontrer qu'il existe une interaction non-triviale entre les paramètres h et β qui donne lieu à un point critique, $h_c(\beta)$, dépendant uniquement de la loi du désordre et de la chaîne de Markov. Si $h > h_c(\beta)$, alors la chaîne de Markov est localisée autour de l'état donné et elle le visite un nombre infinie de fois. Autrement, si $h < h_c(\beta)$, la chaîne de Markov visite l'état donné seulement un nombre fini de fois. Le comportement critique du modèle est donc strictement lié à la structure de $h_c(\beta)$. Un problème très intéressant concerne le comportement du modèle dans la limite du désordre faible. Plus précisément nous cherchons à comprendre le comportement du point critique, $h_c(\beta)$, quand $\beta \to 0$. La réponse dépend de la valeur de α: si elle est plus petite de 1/2, alors il existe une valeur de β sous laquelle le comportement critique du modèle est le même que celui du modèle homogène associé. En revanche, si $\alpha > 1/2$, quelque soit la valeur de β, le désordre perturbe la chaîne de Markov et nous disons qu'il y a pertinence du désordre. Dans

le cas $1/2 < \alpha < 1$, dans la littérature on a des estimations sur l'asymptotique de $h_c(\beta)$ pour $\beta \to 0$ qui ne sont pas précises. Avoir trouvée l'asymptotique précise, c'est à dire un équivalent pour $h_c(\beta)$, représente le résultat le plus important de cette thèse. Précisément on montre que le comportement du modèle d'accrochage de polymère dans la limite du désordre faible est universel et le point critique, opportunément changé d'échelle, converge vers la même quantité donnée par un modèle continu. La preuve est obtenue en utilisant une procédure de coarse-graining, qui généralise les techniques utilisées pour des modèles des polymères proches de celui étudié dans cette thèse.

Sommario

Da un punto di vista chimico e fisico, un polimero è una catena di unità ripetute, chiamate monomeri, quasi identiche nella struttura, ma che possono differire tra loro per il grado di affinità rispetto ad alcuni solventi. Questa caratteristica permette di avere delle interazioni tra il polimero e l'ambiente esterno in cui esso si trova. Nell'ambiente si trova una regione interagente, in maniera positiva o negativa, con il polimero. Questa interazione può avere un effetto sostanziale sulla strutture del polimero, dando luogo a fenomeni di localizzazione e concentrazione ed è dunque possibile osservare l'esistenza di una transizione di fase. Nel caso in cui questa regione è un punto o una linea (oppure un piano o un iper-piano) si parla di modello di pinning – pinning model –, che rappresenta il principale oggetto di studio di questa tesi.

Matematicamente il modello di pinning descrive il comportamento di una catena di Markov in interazione con uno suo stato dato. Questa interazione può attirare o respingere il cammino della catena di Markov con una forza modulata da due parametri, h e β. Quando $\beta = 0$ si parla di modello omogeneo, che è completamente risolubile. Il modello disordinato, che corrisponde a $\beta > 0$, è matematicamente più interessante. In questo caso l'interazione dipende da una sorgente di aleatorietà esterna, indipendente dalla catena di Markov, chiamata disordine. L'interazione è realizzata modificando la legge originale della catena di Markov attraverso una misura di Gibbs (dipendente dal disordine, h e β), cambiando la probabilità di una traiettoria data. L'obiettivo principale è studiare e comprendere la struttura delle traiettorie tipiche della catena di Markov rispetto a questa nuova probabilità.

Modello di pinning con disordine a code pesanti: il primo lavoro di ricerca di questa tesi riguarda il modello di pinning in cui si considera un disordine a code pesanti e il tempo di ritorno della catena di Markov avente una distribuzione sotto-esponenziale. Noi dimostriamo che l'insieme dei tempi in cui la catena di Markov visita lo stato dato, opportunamente riscalato, converge in legge verso un insieme limite, dipendente unicamente dal disordine. Dimostriamo inoltre che esiste una transizione di fase con un punto critico aleatorio, $\beta_c(h)$, sotto il quale l'insieme limite è banale. Questo lavoro ha interessanti connessioni con un altro modello di polimeri molto studiato: il modello di polimero diretto in ambiente aleatorio con code pesanti.

Comportamento critico del modello di pinning nel limite del disordine debole: In questo secondo lavoro consideriamo il modello di pinning con un disordine a code leggere e il tempo di ritorno della catena di Markov con una distribuzione a code polinomiali con esponente caratterizzato da $\alpha > 0$. Sotto queste ipotesi si può dimostrare che esiste un'interazione non banale tra i parametri h e β che dà origine a un punto critico, $h_c(\beta)$, dipendente unicamente dalle leggi del disordine e della catena di Markov. Se $h > h_c(\beta)$, allora la catena di Markov è localizzata attorno allo stato dato e lo visita un numero infinito di volte. Altrimenti, se $h < h_c(\beta)$, la catena di Markov visita lo stato dato solamente un numero finito di volte. Il comportamento critico del modello è dunque profondamente legato alla struttura di $h_c(\beta)$. Un problema molto interessante riguarda il comportamento del modello nel limite del disordine debole: più precisamente vogliamo comprendere il comportamento del punto critico, $h_c(\beta)$, quando $\beta \to 0$. La risposta dipende dal valore di α: se è più piccolo di $1/2$,

allora esiste un valore di β sotto il quale il comportamento critico del modello sarà lo stesso del modello omogeneo associato. Altrimenti, se $\alpha > 1/2$, qualunque sia il valore di β il disordine perturba la catena di Markov e diciamo che il disordine è rilevante. Nel caso $1/2 < \alpha < 1$, in letteratura non esistono stime precise sull'asintotica di $h_c(\beta)$ quando $\beta \to 0$. Aver trovato l'asintotica precisa, ovvero un equivalente per $h_c(\beta)$, rappresenta il risultato più importante di questa tesi. Precisamente dimostriamo che il comportamento del modello di pinning nel limite del disordine debole è universale e il punto critico, opportunamente riscalato, converge verso le rispettive quantità date da un modello continuo. La dimostrazione è ottenuta utilizzando una procedura di coarse-graining, che generalizza le tecniche utilizzate per dei modelli di polimero simili a quello studiato in questa tesi.

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Contents

Polymères aléatoires

1

Le sujet principal de cette thèse concerne les *polymères aléatoires*. Un polymère est une longue molécule linéaire formée par une chaîne d'unités répétées appelées monomères. La définition moderne de polymère a été proposée en 1920 par le chimiste Hermann Staudinger (Prix Nobel de chimie en 1953), qui démontra pour la première fois l'existence de macromolécules organisées dans une structure de chaîne linéaire. Dans la définition IUPAC (International Union of Pure and Applied Chemistry) actuelle [55], un polymère est une substance composée de macromolécules. Une macromolécule est une molécule de masse moléculaire relative élevée et sa structure est essentiellement donnée par la répétition de molécules avec une petite masse moléculaire relative. Ces unités sont appelées *monomères*.

On peut classifier les polymères dans deux grandes familles:

- **homo-polymères** Un polymère avec un seul type de monomères. Les exemples communs sont donnés par les matières plastiques, comme le polyethylene terephthalate (utilisé dans la fabrication des bouteilles en plastique).
- **co-polymère** Un polymère avec plusieurs types de monomères. Des exemples, l'ADN et l'ARN.

L'organisation dans l'espace du polymère est très complexe, avec des auto-interactions entre différents portions de la chaîne et des interactions externes avec l'environnement dans lequel le polymère se trouve. L'interaction avec l'environnement dépend du degré d'affinité de chaque monomère et de leur position dans la chaîne. Un exemple de cette complexité est évidente dans la *Bovine spongiform encephalopathy* (BSE), communément appelé maladie de la *vache folle*. Dans cette maladie neurodégénérative des protéines situées dans les cellules neuronales changent leur structure géométrique. Ce changement conduit à une modification de leur propriétés chimiques qui force une agrégation non-naturel, induisant la mort des cellules neuronales.

Plusieurs efforts de recherche sont concentrés sur la description et la prédiction du comportement d'un polymère en interaction avec un environnement externe. Ce sujet a dépassé les sciences expérimentales, devenant un important sujet de recherche en physique théorique et en mathématiques, introduisant le concept de *polymère abstrait*. L'analyse rigoureuse des propriétés mathématiques des polymères abstraits est un domaine de recherche actif dans la physique mathématique et des progrès importants ont été réalisés dans les dernières années, voir, e.g., les monographies [31, 21, 44, 45] et l'article de revue [38].

Pour comprendre le concept de polymère abstrait on peut essayer à trouver une réponse à la (fondamentale) question

Quelle est la façon mathématique la plus simple pour modéliser un polymère?

L'idée est d'introduire un modèle simplifié, qui peut décrire la nature essentielle du phénomène observé, conduisant à la compréhension des mécanismes de base. De plus, si on veut faire des mathématiques, on ne peut pas être trop loin de modèles super-simplifiés. Nous approchons le problème en regardant le polymère en interaction avec l'environnement externe comme un système (la chaîne du polymère) perturbé par un champ externe (l'environnement) afin d'utiliser les principes de la mécanique statistique pour comprendre les effets de l'environnement sur le polymère. Ces principes suggèrent d'introduire une classe de modèles mathématiques basés sur les marches aléatoires et plus précisément les *marches aléatoires auto-évitant*: un incrément correspond à un monomère du polymère et une réalisation de la marche représente une configuration spatiale (ou trajectoire) du polymère. La contrainte "auto-évitante" décrit le fait que deux monomères ne peuvent pas occuper la même position dans l'espace. En ce sens un polymère abstrait est une trajectoire de marche aléatoire sur un graphe donné, comme $\mathbb{Z}^2$, $\mathbb{Z}^3$ ou, plus généralement, $\mathbb{Z}^d$. Les marches aléatoires auto-évitante sont des modèles extrêmement difficiles et ils présentent encore de nombreuses questions ouvertes [64]. Les modèles de polymère beaucoup plus traitables, satisfaisant la contrainte auto-évitante, sont basés sur les *marches aléatoires dirigées*. Ces sont tout simplement des marche aléatoires dans lesquels une composante est déterministe et strictement croissante, i.e., $(n, S_n)_{n\in\mathbb{N}}$, où $S = (S_n)_{n\in\mathbb{N}}$ est une marche aléatoire sur $\mathbb{Z}^{d-1}$. Ce choix permet d'éviter certains problèmes techniques, comme la présence des spirales – dead-loops – dans les trajectoires de la marche aléatoire.

Une fois introduit le polymère, il faut définir l'environnement et sa manière d'interagir avec le polymère. Considérons une situation réelle, dans laquelle il y a un polymère en interaction avec une membrane pénétrable (ou impénétrable). Imaginons que chaque monomère a un degré donné d'affinité avec la membrane, qui peut être positif ou négatif. Une affinité positive correspond au fait que le monomère est attiré par la membrane et donc il aura tendance à se localiser autour d'elle. Au contraire, une affinité négative signifie qu'il est repoussé. Pour comprendre si la chaîne est localisé ou de-localisé autour de la membrane, nous devons connaître la fraction de monomères avec des interactions positives/négatives et leur placement dans la chaîne. Pour construire un modèle pour décrire ces interactions on peut idéaliser la membrane comme une région donnée de l'espace, par exemple une ligne ou un plan, et à chaque fois que la marche aléatoire traverse cette région d'interaction, elle reçoit une récompense/pénalité donnée par le site de contact. Ces récompenses/pénalités perturbent la trajectoire de marche aléatoire, donnant lieu à des phénomènes de localisation/de-localisation. Mathématiquement, la localisation signifie que la marche aléatoire $(n, S_n)_{n\in\mathbb{N}}$ a un densité d'intersections positives avec la membrane, voir Figure 2.1.

Pour étudier les interactions entre le polymère et la membrane il suffit de connaître les sites de contact entre eux. Nous pouvons aller au-delà des modèles de marches aléatoires dirigées en considérant des processus de contact plus généraux, qui sont les *processus de renouvellement* [6, 37] et le modèle de polymère associé est appelé *modèle d'accrochage de polymère* (*pinning model*). Cette généralisation permet, par exemple, de décrire une grande classe d'interactions entre différents polymères, comme l'ADN. L'ADN est une longue molécule composée d'une double-hélice de deux polymères collés ensemble par des liaisons chimiques. Dans la Section 2.2 nous décrivons un modèle basé sur le processus de renouvellement qui décrit la dénaturation de l'ADN, i.e., le processus chimique qui permet la séparation de la double-hélice. Typiquement la dénaturation de l'ADN dépende de certains facteurs extérieurs, comme la température: des températures élevées induisent le processus de dénaturation de l'ADN, tandis que des basses températures favorisent une situation où la double-hélice est liée. De plus, il existe une température critique au-dessus de laquelle on a la dénaturation de l'ADN. Pour introduire ce modèle, nous idéalisons l'ADN comme une alternance de chemins rectilignes, correspondant aux parties où la double-hélice est liée, et des boucles, où la double-hélice est ouverte. Un processus de renouvellement bien choisi donne les points où l'hélice se divise, donnant lieu à les boucles. La formation d'une boucle est réglée par un coefficient T qui décrit la température du système. Les interactions entre les différentes parties de la chaîne et, plus généralement, tous les détails concernant l'ADN réel, comme la composition chimique, la torsion, sont ignorés. Ce modèle a été introduit par Poland et Scheraga [71] et il est un des premiers modèles d'ADN qui ont fourni l'existence d'une température critique T_c qui divise le régime liée de celui dénaturée.

1.1 Modèle d'accrochage de polymère

Le modèle d'accrochage de polymère représente le sujet central de cette thèse. Dans ce modèle on fixe une marche aléatoire $S = (S_n)_{n\in\mathbb{N}} \subset \mathbb{Z}$ et un nombre $N \in \mathbb{N}$. Une trajectoire du polymère abstrait est donnée par une interpolation linéaire des points $(n, S_n)_{n\leq N}$, qui est le graphe d'une marche aléatoire dirigée de longueur N. Nous fixons une région d'interaction dans l'espace, comme une ligne, et dans le cas le plus simple (le *homogène*), cf. Section 1.2, à chaque fois que la marche aléatoire visite cette région, elle reçoit une récompense constante, positive ou négative, $h \in \mathbb{R}$, qui attire ou repousse la marche aléatoire de la région. On peut penser que si la récompense est positive, alors le polymère a une affinité positive avec cette région.

Si la région est précisément l'axe x, le fait que $(n, S_n)_{n\in\mathbb{N}}$ touche la ligne équivaut à considérer la seule marche aléatoire $(S_n)_{n\in\mathbb{N}}$ en lui donnant une récompense/pénalité à chaque fois qu'elle visite l'état 0. Si nous choisissons une ligne horizontale différente, alors nous changeons l'état avec lequel la marche aléatoire a une interaction privilégiée. Les récompenses/pénalités sont données en changeant la probabilité d'une trajectoire donnée jusqu'à N: à chaque $n \leq N$, nous donnons un poids exponentiel $\exp(h)$, $h \in \mathbb{R}$ à la probabilité que $S_n = 0$. Par conséquent des valeurs positives de h poussent les chemins de la marche aléatoire à visiter souvent 0, tandis que les valeurs négatives de h découragent ces visites. Une fois fixée la marche aléatoire S, il est possible de prouver qu'il existe une valeur critique, h_c tel que, pour $h > h_c$, elle devient récurrent positive, en visitant 0 un nombre infinie de fois, et pour $h < h_c$ la marche aléatoire est transiente, en visitant 0 au plus un numéro fini de fois. Dans le langage des polymères cette discussion signifie qu'il existe une valeur critique de h_c qui divise un comportement localisé d'un de-localisé. La valeur critique h_c est déterminée par la loi originelle de la marche aléatoire: par exemple, si S est le marche aléatoire simple, $h_c = 0$, sinon, si S est une marche aléatoire simple non-symétrique, i.e., $P(X_1 = 1) = p$, $P(X_1 = -1) = 1 - p$, avec $p \neq 1/2$, alors $h_c > 0$ et sa valeur dépend de la différence entre p et $1/2$.

Le cas dans lequel les récompenses ne sont pas homogènes est le plus difficile et mathématiquement le plus intéressant. Il est appelé modèle d'accrochage de polymère *désordonné*, nous parlerons dans la Section 1.3. Dans ce cas, nous introduisons un *désordre* $\omega = (\omega_i)_{i\in\mathbb{N}}$ et les récompenses seront sous la forme $\exp(\beta\omega_n + h)$, où $\beta \in \mathbb{R}_+$. L'idée est que ω_n représente l'affinité entre la région d'interaction et le n-ième monomère. Lassez-nous remarquer que le modèle désordonné est une perturbation du modèle homogène, et la perturbation dépend d'un facteur β. Nous générons le désordre d'une manière aléatoire, i.e., le désordre est une réalisation *gelée* d'une suite aléatoire $\omega = (\omega_i)_{i\in\mathbb{N}}$. Une réalisation donnée correspond à un polymère spécifique et la loi de la séquence aléatoire décrit une famille de polymères. Différents choix de la loi du désordre conduisent à des comportements différents du modèle d'accrochage de polymère. Le cas le plus simple est quand le désordre décrit seulement si un monomère a une affinité positive ou négative avec la région d'interaction – l'état 0 –, i.e., quand $\omega_i \in \{-1, +1\}$. Un autre cas très important est le cas Gaussien, où nous précisons aussi le degré d'affinité de chaque monomère. En général, les cas les plus étudiés ne vont pas plus loin que ces deux [44, 45, 31]. Dans cette thèse, nous concentrons notre attention sur le cas où le désordre est très hétérogène, ce qui signifie mathématiquement que la fonction de distribution de ω_i a une queue lourde, cf. Section 3.1.

Pour comprendre le comportement du modèle d'accrochage de polymère désordonné, nous introduisons le concept de compétition *énergie-entropie*: l'énergie est donnée par la somme totale des récompenses, et l'entropie est le coût associé à la configuration spatiale du polymère: plus une configuration est atypique, plus le coût de l'entropie est élevé. De cette manière, les configurations atypiques peuvent devenir typique pour le polymère si l'énergie gagnée bat le coût entropique associé. Si on suppose que le désordre a des moments exponentiels finis, nous pouvons prouver l'existence d'un point critique $h_c(\beta)$, qui sépare le comportement récurrent positif du comportement transiente de la marche aléatoire. Selon les observations qualitatives faites ci-dessus, si β est grand, alors $h_c(\beta)$ sera différent du point critique d'un modèle homogène. En revanche, si β est très petit, alors la présence du désordre et son influence sur le point critique $h_c(\beta)$ dépend de la structure de la marche aléatoire.

Quand l'introduction d'une quantité arbitrairement petite de désordre modifie le point critique nous parlons de *pertinence du désordre* et dans ce cas un problème ouvert et intéressant consiste à trouver l'asymptotique exacte de $h_c(\beta)$ quand $\beta \to 0$. Le résultat le plus important de cette thèse est la solution à ce problème, cf. Section 3.2.

1.2 Modèle homogène

Pour introduire le modèle d'accrochage de polymère homogène, nous considérons une marche aléatoire simple $(S = (S_n)_{n\in\mathbb{N}}, \mathrm{P})$ sur $\mathbb{Z}$, i.e. $S_n = \sum_{i=1}^n X_i$ où $X_i \in \{-1, 1\}$ et $\mathrm{P}(X_i = -1) = \mathrm{P}(X_i = 1) = 1/2$. La marche aléatoire dirigée associée est le processus $(n, S_n)_{n\in\mathbb{N}}$, qui décrit la configuration du polymère dans l'espace. Dans ce modèle homogène pour $N \in \mathbb{N}$ fixé nous modifions la loi de la marche aléatoire à lui donnant une récompense/pénalité à chaque fois qu'elle visite 0 avant N. Plus précisément, S_n peut visiter 0 seulement quand n est pair, donc pour chaque $N \in 2\mathbb{N}$ nous introduisons la famille de probabilités $\mathrm{P}_{N,h}$ indexes par $h \in \mathbb{R}$ définie comme

$$\mathrm{P}_{N,h}(S) = \frac{1}{\mathrm{Z}_h(N)} \left[\exp\left(h \sum_{n=1}^{N} \mathbb{1}_{S_n=0}\right) \mathbb{1}_{S_N=0}\right] \mathrm{P}(S) \tag{1.1}$$

La constante de normalisation $\mathrm{Z}_h(N)$ est appelée *fonction de partition*. Remarquons que si on permet à X_i de prendre la valeur 0, i.e. $X_i \in \{-1, 0, 1\}$, alors la restriction $N \in 2\mathbb{N}$ n'est pas nécessaire.

Pour définir (1.1) il suffi de connaître la sequence des temps $\tau = \{\tau_0 = 0, \tau_1, \tau_2, \cdots\}$ auxquels la marche aléatoire visite 0, i.e.,

$$\begin{aligned}
\tau_0 &= 0,\\
\tau_k &= \inf\{i > \tau_{k-1} \ : \ S_i = 0\}.
\end{aligned}$$

Pour la propriété de Markov, la suite $(\tau_i - \tau_{i-1})_{i\in\mathbb{N}}$ est i.i.d. sous P et un tel processus est appelé *processus de renouvellement* [6, 37]. La loi du processus de renouvellement est caractérisée par la loi de τ_1

$$K(n) = \mathrm{P}(\tau_1 = n). \tag{1.2}$$

Les exemples plus généraux de processus de renouvellement sont fournis par le temps de retour à zéro des chaînes de Markov discrètes. D'autre part, si nous considérons un processus de renouvellement τ, alors on peut vérifier que $A_n = n - \sup\{\tau_k \ : \ \tau_k \leq n\}$ est une chaîne de Markov avec son ensemble de niveau zéro, $\{n \in \mathbb{N}_0 \ : \ A_n = 0\}$, qui est donné par τ. Cette dualité entre les processus de renouvellement et les chaînes de Markov est représentée en Figure 2.2.

Remarquons que dans le cas de la marche aléatoire simple et symétrique la formule de Stirling fournit l'asymptotique de $K(2n)$, voir e.g. [44, Appendix A.6]:

$$K(2n) = \mathrm{P}(\tau_1 = 2n) = \mathrm{P}\left(S_{2n} = 0, S_1 \neq 0, \cdots, S_{2n-1} \neq 0\right) \underset{n\to\infty}{\sim} \sqrt{\frac{1}{4\pi}} \frac{1}{n^{3/2}}.$$

En particulier on en déduit que l'état 0 est récurrent pour la marche aléatoire simple et symétrique. D'autre part, pour une marche aléatoire simple mais non-symétrique, i.e., $\mathrm{P}(X_1 = 1) = p$, $\mathrm{P}(X_1 = -1) = 1 - p$, $p \neq 1/2$, le même genre d'estimation fournit $\mathrm{P}(\tau_1 < \infty) < 1$, ce qui signifie que la marche aléatoire est transiente. Ceci motive le fait que pour un processus de renouvellement général, $K(\cdot)$ est une probabilité sur $\mathbb{N} \cup \{\infty\}$, avec $K(\infty) := 1 - \sum_{n\in\mathbb{N}} K(n) \geq 0$. À Chaque fois que $K(\infty) > 0$, on dit que le processus de renouvellement τ est *terminant*. Ceci est équivalent à dire que p.s. τ est donné par un nombre fini de points. La classe de processus de renouvellement considérée dans cette section généralise le cas de la marche aléatoire simple. Cette famille est indexée par un exposant $\alpha > 0$, qui contrôle le comportement polynomial de $K(\cdot)$

$$K(n) \sim \frac{L(n)}{n^{1+\alpha}}, \quad n \to \infty, \tag{1.3}$$

où $L(\cdot)$ est une function à variation lente. Nous rappelons que $L : \mathbb{R}_+ \to \mathbb{R}_+$ est une function à variation lente si et seulement si pour $x \in \mathbb{R}_+$ on a que

$$\lim_{t\to\infty} \frac{L(tx)}{L(t)} = 1.$$

Des exemples importants sont fournis par les fonctions asymptotiquement équivalentes à une constante— $L(t) \sim c_K$ — ou alors à une puissance logarithmique — $L(t) \sim (\log t)^\gamma$, $\gamma \in \mathbb{R}$ — cf. [14].

Remarque 1.1. Un autre choix intéressant pour $K(\cdot)$ est le cas sous-exponentiel, $K(n) \sim L(n)e^{-n^\gamma}$, quand $n \to \infty$, $\gamma \in \mathbb{R}$. Récemment cette classe de processus de renouvellement a commencé à être étudiée dans plusieurs travaux de recherche, [62, 78], et elle est le focus du premier résultat de cette thèse, cf. Section 3.1.

Si nous utilisons le processus de renouvellement au lieu de la marche aléatoire, dans la définition du modèle d'accrochage de polymère (1.1) il faut remplacer la marche aléatoire S par le processus de renouvellements τ, obtenant

$$\mathrm{P}_{N,h}(\tau) = \frac{1}{\mathrm{Z}_h(N)} \left[\exp\left(h \sum_{n=1}^{N-1} \mathbb{1}_{n\in\tau} \right) \mathbb{1}_{N\in\tau} \right] \mathrm{P}(\tau), \tag{1.4}$$

où la notation $n \in \tau$ signifie qu'il existe $j \in \mathbb{N}$ tel que $\tau_j = n$. Pour des raisons technique on suppose que

$$K(n) > 0, \quad \forall n \in \mathbb{N}, \tag{1.5}$$

et donc que $\mathrm{P}_{N,h}$ est bien définie pour chaque $N \in \mathbb{N}$. Tous le types de périodicité, i.e., $K(n) \neq 0$ seulement sur $\ell\mathbb{N}$ pour quelque $\ell > 1$, peuvent être étudiés en restreignant la définition (1.4) à $N \in \ell\mathbb{N}$.

Un intérêt du modèle homogène vient du fait que la function de partition est complètement explicite:

$$\mathrm{Z}_h(N) = \sum_{k=1}^{N} e^{hk} \mathrm{P}\left(\sum_{n=1}^{N} \mathbb{1}_{n\in\tau} = k\,,\, N \in \tau, \right) = \sum_{k=1}^{N} e^{hk} \sum_{\ell\in\mathbb{N}^k, |\ell|=N} \prod_{j=1}^{k} K(\ell_j),$$

où $|\ell| = \sum_{i=1}^{k} \ell_i$. Donc

$$\mathrm{Z}_h(N) = \sum_{k=1}^{N} \sum_{\ell\in\mathbb{N}^k, |\ell|=N} \prod_{j=1}^{k} e^h K(\ell_j). \tag{1.6}$$

Si $h < -\log(1 - K(\infty))$, alors le membre de droite est une mesure de probabilité et on a que

$$\mathrm{Z}_h(N) = \mathrm{P}\left(N \in \tau^{(h)}\right), \tag{1.7}$$

où $\tau^{(h)}$ est un processus de renouvellement terminant avec $\mathrm{P}\left(\tau_1^{(h)} = n\right) = e^h K(n)$. D'autre part $h > -\log(1 - K(\infty))$ empêche $e^h K(n)$ d'être une mesure de probabilité sur $\mathbb{N} \cup \{\infty\}$, donc nous allons introduire un facteur de normalisation, appelé *énergie libre* $F : \mathbb{R} \to \mathbb{R}_+$, définie comme l'unique solution de l'équation

$$\sum_n \exp(-F(h)n + h)K(n) = 1, \tag{1.8}$$

quand la solution existe, ce qui correspond au cas $h \geq -\log(1 - K(\infty))$. Sinon, on définit $F(h) := 0$ pour $h < -\log(1 - K(\infty))$. Ce choix est motivé par le fait que $F(-\log(1 - K(\infty))) = 0$.

Remarquons que quand la solution existe, alors elle est unique parce-que $x \mapsto \sum_n \exp(-nx)K(n)$ est strictement croissante, donc $F(h) > 0$ pour chaque $h > -\log(1 - K(\infty))$. Ceci permet d'avoir une forme générale de la fonction de partition:

$$Z_h(N) = e^{NF(h)} P\left(N \in \tau^{(h)}\right), \tag{1.9}$$

où $\tau^{(h)}$ est un processus de renouvellement tel que

$$P\left(\tau_1^{(h)} = n\right) = \exp(-F(h)n + h)K(n). \tag{1.10}$$

Notons que si $h < -\log(1 - K(\infty))$, alors (1.9) est exactement (1.7).

Pour résumer, le modèle homogène $P_{N,h}$ est toujours la loi d'un processus de renouvellement conditionné à visiter N et (1.4) peut être écrit de la manière suivante: pour chaque $0 \le \ell_1 < \cdots < \ell_n = N$, $\ell_i \in \mathbb{N}$

$$P_{N,h}(\tau_1 = \ell_1, \cdots, \tau_n = \ell_n) = P\left(\tau_1^{(h)} = \ell_1, \cdots, \tau_n^{(h)} = \ell_n \middle| N \in \tau^{(h)}\right), \tag{1.11}$$

où $\tau^{(h)}$ est le processus de renouvellement introduit en (1.9). De plus il existe une valeur critique de h

$$h_c = -\log(1 - K(\infty)) \tag{1.12}$$

telle que, si $h < h_c$, alors τ est terminant, et si $h > h_c$ non-terminant. Nous pouvons aller au-delà de ce résultat, en donnant une estimation quantitative sur la fraction de points de τ dans $[0, N]$, i.e., $\frac{1}{N}\sum_{n=1}^{N} \mathbb{1}_{n\in\tau}$: si $h > h_c$, alors la loi de τ_1 est $e^{-F(h)n+h}K(n)$, qui a une espérance finie, et donc la distance entre deux points consécutifs est finite. On peut conclure que $\sharp\{\tau \cap [0, N]\} \approx m_h N$, pour quelque $m_h > 0$. D'autre part si $h < h_c$, alors le processus de renouvellement est terminant, et donc il n'y a qu'un nombre fini de points dans $\mathbb{N}$. Ceci suggère que $\sharp\{\tau \cap [0, N]\} = o(N)$, quand $N \to \infty$. Ce résultat peut être rendu rigoureux: nous observons que

$$E_{N,h}\left[\frac{1}{N}\sum_{n=1}^{N} \mathbb{1}_{n\in\tau}\right] = \frac{d}{dh}\frac{1}{N}\log Z_h(N) \tag{1.13}$$

et que

$$F(h) = \lim_{N\to\infty} \frac{1}{N}\log Z_h(N). \tag{1.14}$$

Un argument de convexité assure que nous pouvons échanger la dérivée par rapport à h avec la limite $N \to \infty$, pour obtenir

$$F'(h) = \lim_{N\to\infty} \frac{d}{dh}\frac{1}{N}\log Z_h(N) = \lim_{N\to\infty} E_{N,h}\left[\frac{1}{N}\sum_{n=1}^{N} \mathbb{1}_{n\in\tau}\right]. \tag{1.15}$$

En particulier $F'(h) > 0$ si $h > -\log(1 - K(\infty))$, et il est égal à 0 si $h < -\log(1 - K(\infty))$.

Remarque 1.2. Pour démontrer (1.14), nous considérons $Z_h(N)$ écrit comme en (1.9) et nous montrons que si $h \in \mathbb{R}$, alors la function $P\left(N \in \tau^{(h)}\right)$ décroît polynomialement quand $N \to \infty$. Ce résultat est conséquence du Théorème (2.82) et (2.85). Pour une preuve détaillée voir, e.g., [44, Proposition 1.1].

Ce résultat peut être formulé comme la convergence en probabilité de $\frac{1}{N}\sum_{n=1}^{N} \mathbb{1}_{n\in\tau}$ vers $F'(h)$, cf. [45, Proposition 2.9]

Théorème 1.1. *Pour tout $\varepsilon > 0$*

$$\lim_{N\to\infty} P_{N,h}\left(\left|\frac{1}{N}\sum_{n=1}^{N} \mathbb{1}_{n\in\tau} - F'(h)\right| \le \varepsilon\right) = 1. \tag{1.16}$$

De plus $F'(h) = 0$ si $h < -\log(1 - K(\infty))$, tandis que, si $h > -\log(1 - K(\infty))$, $F'(h) > 0$.

Une fois que nous avons identifié la valeur critique h_c, qui définit la transition de phase pour le modèle homogène, nous pouvons considérer l'interaction entre cette transition de phase et le comportement critique de l'énergie libre, c'est-à-dire la régularité de $F(h)$ quand $h \downarrow h_c$. Dans ce but nous regardons le comportement asymptotique de $\Psi(x) = 1 - \sum_{n\in\mathbb{N}} e^{-xn}K(n)$ quand $x \to 0$. Détaillons ce calcul dans le cas où $K(\cdot)$ est une mesure de probabilité sur $\mathbb{N}$ satisfaisant (1.3), avec $L(n)$ égal à une constante c_K, et (1.5). Dans ce cas $h_c = 0$ et

$$\begin{aligned}\Psi(x) = 1 - \sum_{n=1}^{\infty} e^{-xn}K(n) = 1 - \sum_{n=1}^{\infty} e^{-xn}\left(\sum_{j\geq n} K(j) - \sum_{j\geq n+1} K(j)\right)\\ =\left(\sum_{j\geq 1} K(j) + \sum_{n=1}^{\infty} e^{-xn} \sum_{j\geq n+1} K(j)\right) - \sum_{n=1}^{\infty} e^{-xn}\sum_{j\geq n} K(j)\\ =(1-e^{-x})\sum_{n=0}^{\infty}\left(\sum_{j\geq n+1} K(j)\right)e^{-xn}.\end{aligned}$$

Si $\alpha > 1$ on a immédiatement

$$\Psi(x) \underset{x\to 0}{\sim} xE\left[\tau_1\right], \tag{1.17}$$

et si $\alpha \in (0,1)$ une approximation en sommes de Riemann sum donne

$$\Psi(x) \underset{x\to 0}{\sim} x\sum_n \frac{c_K}{\alpha n^{\alpha}} e^{-xn} \underset{x\to 0}{\sim} \frac{c_K\, x^{\alpha}}{\alpha} \int_0^{\infty} s^{-\alpha} e^{-s} ds = c_K \frac{\Gamma(1-\alpha)}{\alpha} x^{\alpha}. \tag{1.18}$$

En rappelant que $\Psi(F(h)) = 1 - e^h$, et en inversant les asymptotique dans les expressions (1.17) et (1.18), on obtiene le comportement critique de $F(h)$. Le résultat précis et général est dans [44, Theorem 2.1]:

Théorème 1.2. *Pour tout $\alpha \geq 0$ et $L(\cdot)$ satisfaisant* (1.3)*, il existe une unique fonction à variation lente $\hat{L}(\cdot)$ telle que*

$$F(h) \overset{h\searrow h_c}{\sim} (h-h_c)^{1/\min\{\alpha,1\}}\hat{L}\left(1/(h-h_c)\right). \tag{1.19}$$

En particulier $\hat{L}(\cdot)$ est une constante si $\alpha > 1$.

Comme conséquence de ce théorème nous avons que si $\alpha > 1$, alors $F(h)$ n'est pas C^1 au point critique et nous disons que la transition est du premier ordre. En générale on dit que le modèle présente une transition de phase de ordre k si $F(h)$ est C^{k-1} mais pas C^k au point critique. Le modèle homogène a une transition de phase de ordre k, $k = 2, 3, \cdots$, à $h = h_c$ si $\alpha \in [1/k, 1/(k-1))$.

1.3 Modèle désordonné

Le *Modèle d'accrochage de polymère désordonné* est défini comme une perturbation aléatoire du modèle homogène (1.4). Pour chaque i, $i = 1, \cdots, N$, nous remplaçons l'exposant h par $\beta\omega_i + h$, où ω_i est une valeur donnée (gelée), indépendante du processus de renouvellement. La suite $\omega = (\omega_i)_{i\in\mathbb{N}}$ est appelée *désordre du système* et elle est une réalisation d'une sequence aléatoire donnée – *gelée*. Nous notons $\mathbb{P}$ sa loi.

Pour $N \in \mathbb{N}$ nous considérerons la famille de mesures de probabilité $\mathrm{P}^{\omega}_{N,h,\beta}$, indexée par $h \in \mathbb{R}$, $\beta \in \mathbb{R}_+$ définie comme

$$\mathrm{P}^{\omega}_{N,h,\beta}(\tau) = \frac{1}{\mathrm{Z}^{\omega}_{\beta,h}(N)}\left[\exp\left(\sum_{n=1}^{N}(\beta\omega_n + h)\mathbb{1}_{N\in\tau}\right)\mathbb{1}_{n\in\tau}\right]\mathrm{P}(\tau). \tag{1.20}$$

Dés lors que ω est une réalisation fixée, nous appelons (1.22) le modèle *gelé* (quenched).

Nous supposons que la suite aléatoire ω est i.i.d. avec des moments exponentiels finis. De plus, ω_1 est de moyenne nulle, de variance unitaire et il existe $\beta_0 > 0$ tel que

$$\Lambda(\beta) = \log \mathbb{E}[e^{\beta\omega_1}] < \infty \quad \forall \beta \in (-\beta_0, \beta_0), \quad \mathbb{E}[\omega_1] = 0, \quad \mathbb{V}[\omega_1] = 1. \tag{1.21}$$

Sous ces hypothèses il est utile de faire le changement de paramétre $h \mapsto h - \Lambda(\beta)$, pour normaliser la variable $e^{\beta\omega_1}$ et nous re-définissons (1.20) comme

$$\mathrm{P}^{\omega}_{N,h,\beta}(\tau) = \frac{1}{\mathrm{Z}^{\omega}_{\beta,h}(N)} \left[\exp\left(\sum_{n=1}^{N} (\beta\omega_n - \Lambda(\beta) + h) \mathbb{1}_{n\in\tau} \right) \mathbb{1}_{n\in\tau} \right] \mathrm{P}(\tau). \tag{1.22}$$

Remarque 1.3. L'hypothèse sur les moments exponentiels est très importante pour les résultats que nous allons présenter dans la suite, et des choix différents de désordre fournissent un comportement très différent pour le modèle d'accrochage de polymère. Ceci est, par exemple, le cas de variables aléatoires à queue lourde que nous discutons dans la Section 3.1.

Par analogie avec le modèle homogène nous étudions le comportement du modèle d'accrochage de polymère par l'analyse de l'énergie libre, définie comme

$$F(\beta, h) = \lim_{N\to\infty} \frac{1}{N} \mathbb{E}\left[\log \mathrm{Z}^{\omega}_{\beta,h}(N) \right], \tag{1.23}$$

où la fonction de partition $\mathrm{Z}^{\omega}_{\beta,h}(N)$ a été introduit en (1.22). L'existence de l'énergie libre décent d'un argument de super-additivité. Le résultat peut-être amélioré en utilisant le théorème ergotique de Kingman [58], qui assure la convergence p.s. du membre de droit, c'est-à-dire

$$F(\beta, h) = \lim_{N\to\infty} \frac{1}{N} \log \mathrm{Z}^{\omega}_{\beta,h}(N), \quad \mathbb{P} - \text{a.s.}$$

Voir e.g. [44, Chapter 4].

Pour étudier les propriétés critiques du modèle d'accrochage de polymère désordonné nous commençons par noter que le modèle d'accrochage de polymère homogènes fournit des limites inférieures et supérieures pour l'énergie libre:

$$F(0, h - \Lambda(\beta)) \leq F(\beta, h) \leq F(0, h). \tag{1.24}$$

En particule, on a que $F(\beta, h) \geq 0$, parce-que l'énergie libre du modèle homogène est positive. L'inégalité de droit, $F(\beta, h) \leq F(0, h)$, est une conséquence immédiat de l'inégalité de Jensen et elle est appelée borne *annealed*. L'inégalité de Jensen fournit aussi la preuve de l'inégalité de gauche:

$$\begin{aligned} \log \mathrm{Z}_{\beta,h}(N) &= \log \mathrm{E}_{N,h-\Lambda(\beta)} \left[e^{\sum_{n=1}^{N} \beta\omega_n \mathbb{1}_{n\in\mathbb{N}}} \right] + \log \mathrm{Z}_{N,h-\Lambda(\beta)} \\ &\geq \sum_{n=1}^{N} \beta\omega_n \mathrm{P}_{N,h-\Lambda(\beta)}(\tau_1 = N) + \log \mathrm{Z}_{N,h-\Lambda(\beta)}, \end{aligned}$$

où $\mathrm{E}_{N,h-\Lambda(\beta)}$ est l'espérance du modèle homogène de paramètre $h - \Lambda(\beta)$. En prenant l'espérance par rapport au désordre $\mathbb{E}$ on obtient le le résultat, car $\mathbb{E}[\omega_n] = 0$.

Nous remarquons que pour chaque $\beta > 0$ la function $h \mapsto F(\beta, h)$ est monotone et donc le *point critique*

$$h_c(\beta) := \sup\{h \,:\, F(\beta, h) = 0\} \tag{1.25}$$

est bien définie et divise le plan (h, β) à deux régions $\mathcal{L} = \{(\beta, h) : F(\beta, h) > 0\}$ and $\mathcal{D} = \{(\beta, h) : F(\beta, h) = 0\}$. On peut prouver, pour le modèle désordonné, une version analogue du Théorème 1.1

(cf. les monographies [44, 31, 45].):

$$\begin{aligned}
(\beta, h) \in \mathcal{L} \quad &\Leftrightarrow \quad \exists\, m_{\beta,h} > 0 : \forall\, \varepsilon > 0 \quad \mathrm{P}^{\omega}_{N,h,\beta}\left(\left|\frac{1}{N}\sum_{n=1}^{N} \mathbb{1}_{n\in\tau} - m_{\beta,h}\right| \le \varepsilon\right) \underset{N\to\infty}{\overset{\mathbb{P}-\text{probability}}{\rightarrow}} 1, \\
(\beta, h) \in \overset{\circ}{\mathcal{D}} \quad &\Leftrightarrow \quad : \forall\, \varepsilon > 0 \quad \mathrm{P}^{\omega}_{N,h,\beta}\left(\left|\frac{1}{N}\sum_{n=1}^{N} \mathbb{1}_{n\in\tau}\right| \le \varepsilon\right) \underset{N\to\infty}{\overset{\mathbb{P}-\text{probability}}{\rightarrow}} 1.
\end{aligned} \tag{1.26}$$

De plus (1.24) donne

$$h_c(0) \le h_c(\beta) \le h_c(0) + \Lambda(\beta) \tag{1.27}$$

et $h_c(0)$ est la *valeur critique annealed*. Nous rappelons que $h_c(0) = -\log(1 - K(\infty))$. Il est maintenant naturel de se demander si la présence du désordre joue un rôle ou non. Une prédiction heuristique, mais précise, est donnée par le *critère de Harris* [51] et Figure 2.3.

1.3.1 Le critère de Harris

Le critère de Harris a été proposé dans les années 70 pour comprendre si et comment l'introduction d'une petite quantité de désordre dans un modèle homogène peut changer son comportement critique. Dans l'article original [51] seul le *modèle d'Ising dilué* est considéré, mais les idées développées sont très utiles pour comprendre la présence du désordre dans de nombreux modèles désordonnés, parmi lesquels on retrouve le modèle d'accrochage de polymère. Dans ce dernier cas, ce critère a inspiré différentes méthodes pour comprendre quand le désordre ω perturbe le comportement du modèle, c'est-à-dire quand les propriétés critiques du système sont différentes de celles d'un modèle *annealed* [33, 40]. Le modèle annealed est un type particulier de modèle homogène, cf. (1.28). Ce que nous entendons par propriété critique concerne le comportement du modèle à proximité du point critique, i.e., quand $h \approx h_c(\beta)$. En particulier un des nos buts est de décrire comment $F(\beta, h)$ converge vers 0 quand $h \searrow h_c(\beta)$, en comparant l'exposant critique du modèle désordonné avec l'exposant critique du modèle annealed, (qui est le même que celui du modèle homogène, cf. (1.19)). Chaque fois que l'exposant critique du modèle désordonné est différente de celui homogène, même si β est arbitrairement petit, nous disons qu'il y a *pertinence du désordre*.

Dans le cas du modèle d'accrochage de polymère l'analyse de l'exposant critique est strictement liée à l'étude de la structure du point critique $h_c(\beta)$. La réponse finale est qu'il n'y a pas pertinence du désordre si $\alpha < 1/2$ et qu'il y a pertinence du désordre si $\alpha > 1/2$, où α est l'exposant du processus de renouvellement, cf. (1.3), voir Figure 2.3. Le cas $\alpha = 1/2$ est plus compliqué et il est appelé marginale: la pertinence du désordre dépend du choix de la fonction à variation lente L en (1.3). Nous allons expliquer ceci par un argument heuristique inspiré de celui utilisé dans [33, 40] et [45].

Remarque 1.4. À chaque fois que nous étudions le comportement critique du modèle, sans perte de généralité on suppose que le processus de renouvellement est non-terminant: il suffit d'opérer un changement de variable $h \mapsto h + h_c$ dans la définition du modèle d'accrochage de polymère (1.22) et d'utiliser un processus de renouvellement non-terminant τ' tel que $K'(n) = K(n)/(1 - K(\infty))$.

Dans la suite nous supposons que $K(\infty) = 0$, et donc que le point critique est $h_c(0) = 0$.

Nous fixons une valeur de $\beta > 0$ et $h \ge 0$. Nous considérons la fonction de partition *annealed*

$$\mathrm{Z}^{\mathrm{ann}}_{h}(N) := \mathbb{E}\left[\mathrm{Z}^{\omega,\mathtt{f}}_{\beta,h}(N)\right] = \mathbb{E}\mathrm{E}\left[e^{\sum_{n=1}^{N}(\beta\omega_n - \Lambda(\beta) + h)\mathbb{1}_{n\in\tau}}\right] = \mathrm{E}\left[e^{h\sum_{n=1}^{N}\mathbb{1}_{n\in\tau}}\right], \tag{1.28}$$

qui correspond au modèle homogène, cf. Section 1.2. On observe que

$$\frac{\mathrm{Z}^{\omega,\mathtt{f}}_{\beta,h}(N)}{\mathrm{Z}^{\mathrm{ann}}_{h}(N)} = \mathrm{E}^{\mathrm{ann}}_{h,N}\left[\exp\left\{\sum_{n=1}^{N}(\beta\omega_n - \Lambda(\beta))\mathbb{1}_{n\in\tau}\right\}\right], \tag{1.29}$$

où $\mathrm{E}_{h,N}^{\mathrm{ann}}$ est l'espérance du modèle annealed (homogène). Soit

$$\zeta(x) = e^{\beta x - \Lambda(\beta)} - 1, \tag{1.30}$$

et on réécrit (1.29) comme un polynôme

$$\begin{aligned} &\mathrm{E}_{h,N}^{\mathrm{ann}}\left[\exp\left\{\sum_{n=1}^{N}(\beta\omega_n - \Lambda(\beta))\mathbb{1}_{n\in\tau}\right\}\right] = \mathrm{E}_{h,N}^{\mathrm{ann}}\left[\prod_{n=1}^{N}(\zeta(\omega_n)\mathbb{1}_{n\in\tau} + 1)\right] \\ =&1 + \sum_{n=1}^{N}\zeta(\omega_n)\mathrm{P}_{h,N}^{\mathrm{ann}}(n\in\tau) + \cdots \end{aligned} \tag{1.31}$$

Remarquons que $(\zeta(\omega_i))_{i\in\mathbb{N}}$ est une suite i.i.d. de variables aléatoires centrées de variance $\sim \beta^2$ quand $\beta \to 0$. Donc on peut approximer $\zeta(\omega_i)$ par $\beta\tilde{\omega}_i$, où $(\tilde{\omega}_i)_{i\in\mathbb{N}}$ est une sequence de Gaussiennes standard. Donc (1.31) implique que

$$\mathbb{E}\log\frac{Z_{\beta,h}^{\omega,\mathfrak{f}}(N)}{Z_h^{\mathrm{ann}}(N)} \approx -\frac{1}{2}\beta^2\sum_{n=1}^{N}\mathrm{P}_{h,N}^{\mathrm{ann}}(n\in\tau)^2 + \cdots \tag{1.32}$$

Nous notons que si $h > 0$ et n, $N-n$ divergent vers ∞ quand N croit vers ∞, alors

$$\mathrm{P}_{h,N}^{\mathrm{ann}}(n\in\tau) = \frac{Z_h^{\mathrm{ann}}(n)Z_h^{\mathrm{ann}}(N-n)}{Z_h^{\mathrm{ann}}(N)} \overset{(1.9)}{=} \frac{\mathrm{P}\left(n\in\tilde{\tau}^{(h)}\right)\mathrm{P}\left(N-n\in\tilde{\tau}^{(h)}\right)}{\mathrm{P}(N\in\tilde{\tau}^{(h)})} \underset{n,N-n\to\infty}{\sim} \frac{1}{\mathrm{E}\left[\tilde{\tau}^{(h)}\right]}$$

par le Théorème de renouvellement (Renewal Theorem) (2.82). En utilisant (1.15) on a que $\frac{1}{N}\sum_{n=1}^{N}\mathrm{P}_{h,N}^{\mathrm{ann}}(n\in\tau) \underset{N\to\infty}{\sim} F'(0,h)$, donc $F'(0,h) = 1/\mathrm{E}\left[\tilde{\tau}^{(h)}\right]$ par le Théorème de Césaro. Ces estimations suggèrent de remplacer $\mathrm{P}_{h,N}^{\mathrm{ann}}(n\in\tau)$ par $F'(0,h)$ en (1.32), en obtenant

$$F(\beta,h) \approx F(0,h) - \frac{1}{2}\beta^2(F'(0,h))^2 + \cdots. \tag{1.33}$$

En utilisant (1.19) avec $\alpha\in(0,1)$ on a $F(0,h)\approx h^{1/\alpha}$ et donc $F'(0,h)^2 \approx h^{2(1/\alpha-1)}$. Dans le cas $\alpha < 1/2$ on a que $h^{2(1/\alpha-1)}$ est négligeable par rapport à $h^{1/\alpha}$ et donc $F(\beta,h)\approx h^{1/\alpha}$, qui est le comportement du modèle annealed. D'autre part, si $\alpha > 1/2$, alors le deuxième terme dans cette expansion est beaucoup plus grand que le premier. Ceci est un symptôme du fait que quelque chose ne fonctionne pas dans notre développement, e.g. on est en train de développer autour du mauves point. Pour trouver le point correct on note que $F(\beta,h) = 0$, quelque soit $h \le h_c(\beta)$, donc $h_c(\beta)$ devrait être équivalente à la valeur pour laquelle le côté droit de (1.33) devient zéro. Ceci implique que $h_c(\beta) > 0\,(= h_c(0))$ et

$$h_c(\beta) \approx \beta^{\frac{2\alpha}{2\alpha-1}}, \quad \text{pour } \beta \text{ petit.} \tag{1.34}$$

La plupart de ces résultats ont été rendus rigoureux. En particule le fait que le désordre n'est pas pertinent si $\alpha < 1/2$ a été prouvée en [4, 26, 60] et en [49] il a été montré que l'exposant critique du modèle désordonné doit être plus grand de 2. Ceci signifie que si $\alpha > 1/2$, alors l'exposant critique du modèle désordonné est strictement plus grand de celui annealed, qui est égal à $\frac{1}{\alpha}$ cf. (1.19).

Les cas $\alpha \in (1/2, 1)$ a été étudié en profondeur dans différents travaux [3, 32], confirmant (1.34) sans trouver la constant exacte: plus précisément il existe une fonction à variation lente $\tilde{L}_\alpha$ (explicitement déterminée par L et α, voir Remarque 5.2), et une constante $0 < c < \infty$ telle que si $\beta > 0$ est suffisamment petit

$$c^{-1}\tilde{L}_\alpha(\beta^{-1})\beta^{\frac{2\alpha}{2\alpha-1}} \le h_c(\beta) \le c\tilde{L}_\alpha(\beta^{-1})\beta^{\frac{2\alpha}{2\alpha-1}}. \tag{1.35}$$

Le cas marginal (qui inclut la marche aléatoire simple), $\alpha = 1/2$, a été considéré dans plusieurs travaux pendant ces dernières années [4, 46, 47, 77], et il a été résolu récemment en [10]. Dans ce cas, les propriétés critiques du modèle dépendent du choix de la fonction à variation lente L et $h_c(\beta) > 0$ quelque soit β si et seulement si $\sum_n \frac{1}{n\,L(n)^2} = \infty$.

Dans [9] le cas $\alpha > 1$ a été complètement résolu, en trouvant l'asymptotique précise de $h_c(\beta)$ quand $\beta \to 0$:

$$h_c(\beta) \underset{\beta\to 0}{\sim} \frac{\alpha}{2\mathrm{E}\,[\tau_1]\,(1+\alpha)}\beta^2 \tag{1.36}$$

Dans cette thèse — Section 3.2 — nous trouvons l'asymptotique précise de $h_c(\beta)$ quand $\beta \to 0$ et $\alpha \in (1/2, 1)$, en affinant (1.35) et en rendant rigoureuse la déduction de (1.34). En particulier nous démontrons que

$$\lim_{\beta\to 0} \frac{h_c(\beta)}{\beta^{\frac{2\alpha}{2\alpha-1}}\,\tilde{L}_\alpha(\beta^{-1})} = \mathbf{h}_c(1). \tag{1.37}$$

où m_α est une constante universelle dépendant seulement de α et est donnée par le point critique du modèle continu associé, [23, 24], que nous allons introduire dans la Section 2.3.2. Remarquons la valeur *universelle* de (2.54): le comportement asymptotique de $h_c(\beta)$ quand $\beta \to 0$ ne dépend que de la queue de la distribution des temps de retours à zero $K(n) = \mathrm{P}(\tau_1 = n)$, à travers l'exposant $\alpha \in (\frac{1}{2}, 1)$ et la fonction à variation lente L (qui définie $\tilde{L}_\alpha$): tous les autres détails de $K(n)$ deviennent négligeables dans la limite du désordre faible $\beta \to 0$. La même chose vaut pour le désordre ω: toutes les distributions admissibles pour ω_1, cf. (3.18)-(3.19) ci-dessous, ont le même effet sur le comportement critique de $h_c(\beta)$.

Polymers and probability 2

The main focus of this thesis are *random polymers*. A polymer is a long linear molecule formed by a chain of repeating units, called monomers. The modern definition of polymer was proposed in 1920 by the chemist Hermann Staudinger (Nobel Prize for Chemistry in 1953), who first demonstrated the existence of macromolecules organized in a chain structure. In the current IUPAC's (International Union of Pure and Applied Chemistry) definition [55] a *polymer* is a substance composed of macromolecules. A macromolecule is a molecule of high relative molecular mass with structure essentially given by the multiple repetition of units derived from molecules of low relative molecular mass. The constitutional units to the essential structure of a macromolecule are called *monomers*.

We can classify the polymers in two big families

- **homopolymer** A polymer derived from one species of monomers. Common examples are provided by plastic material, like polyethylene terephthalate (used for plastic bottles).
- **copolymer** A polymer derived from more than one species of monomer. Common examples are the DNA and RNA.

The polymer chain is organized in the space in a complex structure, with possible self-interactions between different portions of the chain and also external interactions with the environment in which the polymer is. The interaction with the environment depends on the degree of affinity of each monomer and on their spatial position. An example of this complexity is evident in the *Bovine spongiform encephalopathy* (BSE), commonly known as *mad cow disease*. In this neurodegenerative disease natural proteins situated in neuronal cells change their geometrical structure, leading to a substantial modification of the chemical properties which causes an unnatural aggregation, inducing the death of neuronal cells. Several research efforts are concentrated on describing and predicting the behavior of a polymer in interaction with an external environment. This topic has gone beyond the experimental sciences, becoming an important research subject in theoretical physics and mathematics, leading to the introduction of the definition of *abstract* polymers. The rigorous analysis of the mathematical properties of such abstract polymer is an active research field in statistical physics and important progress has been made in the last years, see e.g. the monographs [31, 21, 44, 45] and the review article [38].

To introduce an abstract polymer we can start from the (fundamental) question:

What is the simplest mathematical way to model a polymer?

The idea is to introduce a simplified model, which may catch the essential nature of the phenomena, leading to the understanding of the basic mechanisms. Even more, if one wants to do mathematics, then one cannot get too far from oversimplified models. We approach the problem by looking at the polymer in interaction with an external environment as a system (the polymer chain) perturbed by an external field (the environment) in order to use the principles of statistical mechanics to understand the effects of the environment on the polymer. These principles suggest to introduce a class of mathematical models based on the random walks, in particular *self-avoiding random walk*: an increment of the random walk is a monomer of the polymer and a realization of the random walk represents a spatial configuration (or trajectory) of the polymer. The self-avoidance constraint describes the fact that two monomers cannot occupy the same place in space. In this sense an *abstract polymer* is a random walk path on a underlying lattice, like $\mathbb{Z}^2$, $\mathbb{Z}^3$ or, more generally, $\mathbb{Z}^d$. The difficulty is that self-avoiding random walks are extremely challenging models and they still present many open questions [64]. Much more treatable polymer models, satisfying the self-avoiding constraint, are based on *directed walk models*. These are simply random walks in which one component is deterministic and strictly increasing, like $(n, S_n)_{n\in\mathbb{N}}$, where $S = (S_n)_{n\in\mathbb{N}}$ is a random walk on $\mathbb{Z}^{d-1}$. This choice allows to avoid some technical problems, like the presence of possible dead-loops in the random-walk path.

Once introduced the polymer, we have to define the environment and how it interacts with the polymer. For this purpose let us consider a real situation in which a polymer is in interaction with a penetrable (or impenetrable) membrane. We can image that each monomer has a given degree of affinity with the membrane, which can be positive or negative. A positive affinity means that the monomer is attracted by the membrane and it will have the tendency to be localized near to the membrane, while negative affinity means that it is pushed away. Of course to understand if the whole chain will be localized or not on the membrane we have to know the fraction of monomers with positive/negative interactions and their placement along the chain. To build up a model to describe this interaction we idealize the membrane as a given region of the space, like a line or a plain, and each time that the random walk crosses this region of interaction we give a reward/penalty to the contact site. These rewards/penalties perturb the random walk trajectory and localization/de-localization phenomena can appear. Mathematically by localization we mean that the random walk trajectory $(n, S_n)_{n\in\mathbb{N}}$ has a positive density intersections with the membrane, see Figure 2.1.

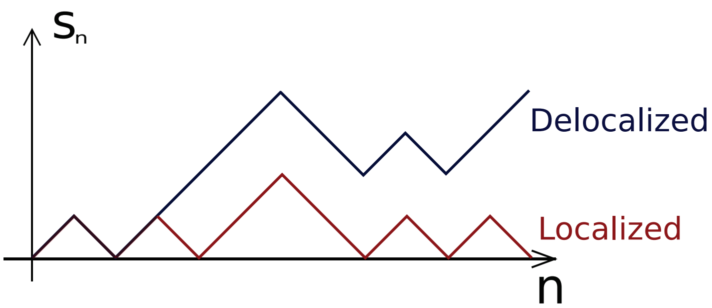

Fig. 2.1: Different behaviors of an abstract polymer in interaction with the x-axis: the polymer can be localized (red), by crossing the x-axis infinitely many times, or de-localized (blue) in which it visits only a finite number of times the region of interaction.

It is clear that to study the interactions between the polymer and the membrane it is enough to know the contact sites between them. We can go beyond directed walk models by considering more general contact site processes, which actually are the *renewal processes* [6, 37] and the associated polymer model is called *pinning model*. This generalization allows to describe a large class of interactions between different polymers, like the DNA. The DNA is a long molecule made by a double-helix of two polymers glued together through chemical bonds. In Section 2.2 we discuss a suitable model based on the renewal process which describes the DNA denaturation, that is the chemical process which allows the separation of the double-helix. Typically the DNA denaturation depends on external factors, like the temperature: high temperatures induce the DNA denaturation process, while low temperatures favor the double-helix to stay bound. In particular there exists a critical temperature above which the DNA denaturation happens. To introduce such model, we idealize the DNA as an alternating sequence of straight paths, corresponding to the stretches where the double-helix is bound, and loops, where the double-helix is broken. A suitable renewal process provides the points in which the double-helix divides forming such loops. The ease to form a loop is tuned by a factor T which describes the temperature of the system. Interactions between different parts of a chain and, more generally, all details regarding real DNA, such as chemical composition, stiffness or torsion, are ignored. This model has been introduced by Poland and Scheraga [71] and it is one of the first DNA models which have provided the existence of a non-trivial critical temperature T_c which divides the bound regime from the denaturated one.

2.1 Pinning model

The main polymer model considered in this thesis is the *pinning model*. In this model we consider a random walk $S = (S_n)_{n\in\mathbb{N}} \subset \mathbb{Z}$, and the *trajectory* of our abstract polymer will be given by the linear interpolation of the points $(n, S_n)_{n\in\mathbb{N}}$, which is the graph of a directed random walk (a random walk trajectory). Then we select a reasonable region of interaction in the space, like a line, and in the simplest case, the *homogeneous* one, cf. Section 2.2, each time the random walk trajectory crosses the line, it receives a positive or negative reward $h \in \mathbb{R}$ which can attract or repel the random walk path from the region. We expect that if the reward is always positive, then the polymer will prefer to stay close to the line, otherwise it goes away. If the region is precisely an horizontal line like the x-axis, saying that $(n, S_n)_{n\in\mathbb{N}}$ crosses the line is equivalent to consider only the random walk $(S_n)_{n\in\mathbb{N}}$ by giving a reward/penalty each time it visits the state 0. If we choose a different horizontal line, then we change the state with which the random walk has a privileged interaction. The rewards/penalties are given by biasing the probability of a given trajectory up to time N: at any time $n \leq N$, we assign an exponential weight $\exp(h)$, $h \in \mathbb{R}$ to the probability that $S_n = 0$. Therefore large positive values of h push the random walk paths to visit often 0, while negative values of h discourage such visits. Once fixed the random walk S, it is possible to prove that there exists a critical value of h_c such that, for $h > h_c$, S becomes positive recurrent, while for $h < h_c$ the random walk is transient, by visiting 0 no more than a finite number of times. In the polymers language this discussion means that there exists a critical value of h_c which divides a localized behavior from a de-localized one. The critical value h_c turns out to be determined by the original transience/recurrence of the random walk, for instance if S is the symmetric simple random walk, then $h_c = 0$, otherwise, if the walk is p-asymmetric, with $p \neq 1/2$, then $h_c > 0$ and its value depends on the discrepancy of p from $1/2$.

The case in which the rewards change along the line is more challenging and it goes under the name of *disordered pinning model*, which we discuss in Section 2.3. In this case we introduce a *disorder* $\omega = (\omega_i)_{i\in\mathbb{N}}$ and the rewards will be in the form $\exp(\beta\omega_n + h)$, where $\beta \in \mathbb{R}_+$. The idea is that ω_n is the affinity with the interaction region of the n-th monomer. Let us stress that the disordered model is nothing but a perturbation of the homogeneous model and such perturbation depends on a factor β. We generate the disorder in a random way, that is the disorder is a *quenched* realization of a random sequence $\omega = (\omega_i)_{i\in\mathbb{N}}$, which corresponds to a specific polymer. The law of such random sequence describes a given family of polymers and different choices of disorder law lead to different behaviors of the pinning model. The simplest case is when the disorder says only if a given monomer has positive or negative affinity with the line, that is $\omega_i \in \{-1, +1\}$. Another important case is the Gaussian one, in which we also specify the degree of affinity of each monomer with the line. In general the most studied cases do not run so far from these two [44, 45, 31]. In this thesis we focus our attention also on the case in which the disorder is very inhomogeneous, which mathematically means that the distribution function of ω_i has heavy tails, see Section 3.1. To understand the behavior of the disordered pinning model we introduce the concept of *energy-entropy* competition: the energy is given by the sum of the total rewards, while the entropy is the cost associated to the spatial configuration of the polymer: the more a configuration is atypical, the higher the entropy cost. In such a way atypical configurations can become typical for the polymer if the energy gained beats the associated entropic cost. If we assume the finiteness of the exponential moments of the disorder, we can prove the existence of a critical point $h_c(\beta)$, which divides the recurrent/transient behavior of the random walk. According to the qualitative observations made above, if β is large, then $h_c(\beta)$ will be different from the critical point of an homogeneous model, while if β is very small, then to understand when the disorder presence influences the critical point $h_c(\beta)$ is not obvious and it depends on the structure of the random walk. When the introduction of the disorder modifies the critical point we say that the disorder is *relevant* and in such case a natural and interesting open problem is to find the exact asymptotics of $h_c(\beta)$ as $\beta \to 0$. The main result of this thesis regards the solution of such problem, see Section 3.2.

2.2 Homogeneous pinning model

To introduce the homogeneous pinning model let us consider a symmetric simple random walk $(S = (S_n)_{n\in\mathbb{N}}, \mathrm{P})$ on $\mathbb{Z}$, i.e. $S_n = \sum_{i=1}^n X_i$ where $X_i \in \{-1, 1\}$, with $\mathrm{P}(X_i = -1) = \mathrm{P}(X_i = 1) = 1/2$. Our directed random walk is the process $(n, S_n)_{n\in\mathbb{N}}$ which describes the spatial configuration of the polymer. In the homogeneous pinning model we fix $N \in \mathbb{N}$ and we modify the law of the random walk by rewarding/penalizing each visit to 0 before N. To be more precise we note that S_n can visit 0 only if n is even, therefore for any $N \in 2\mathbb{N}$ we introduce the family of probabilities $\mathrm{P}_{N,h}$ indexed by $h \in \mathbb{R}$ defined as

$$\mathrm{P}_{N,h}(S) = \frac{1}{\mathrm{Z}_h(N)}\left[\exp\left(h\sum_{n=1}^{N}\mathbb{1}_{S_n=0}\right)\mathbb{1}_{S_N=0}\right]\mathrm{P}(S) \tag{2.1}$$

The normalization constant $\mathrm{Z}_h(N)$ is called *partition function*. Note that if we allow X_i to take also 0-value, i.e. $X_i \in \{-1, 0, 1\}$, then the constraint $N \in 2\mathbb{N}$ is not necessary. It is clear that to define (2.1) it is enough to know the sequence of random times $\tau = \{\tau_0 = 0, \tau_1, \tau_2, \cdots\}$ at which the random walk visits 0

$$\begin{aligned}\tau_0 &= 0,\\ \tau_k &= \inf\{i > \tau_{k-1} \;:\; S_i = 0\}.\end{aligned}$$

By Markov's property the sequence of *inter-arrival times* $(\tau_i - \tau_{i-1})_{i\in\mathbb{N}}$ is i.i.d. under P and such a process is called *renewal process* [6, 37]. The law of the renewal process is characterized by the law of τ_1

$$K(n) = \mathrm{P}(\tau_1 = n). \tag{2.2}$$

More general examples of renewal processes are provided by the return time to zero of discrete Markov chains. On the other hand given a renewal process τ we define $A_n = n - \sup\{\tau_k \;:\; \tau_k \le n\}$ which turns out to be a Markov chain with zero level set given by τ and it is called the backward recurrence time. Such duality between renewal processes and Markov's chains is pictured in Figure 2.2.

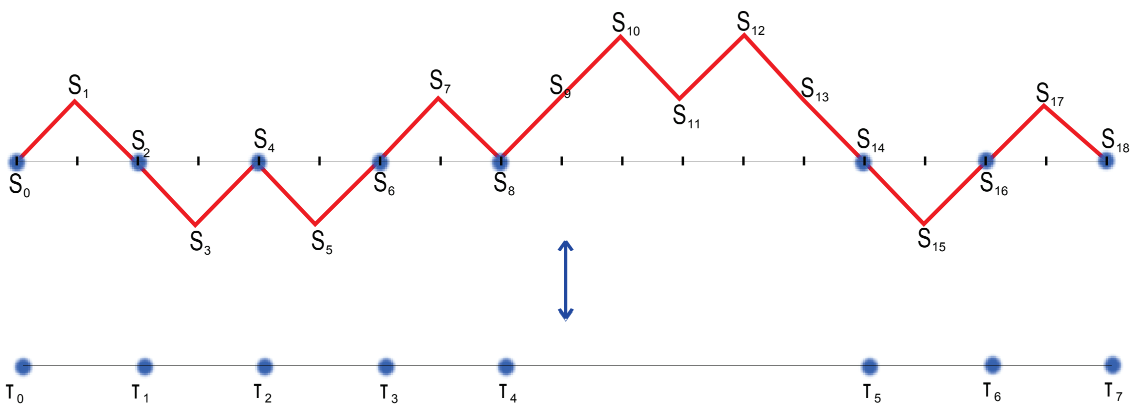

Fig. 2.2: Given a random walk S starting from 0, we can always define a renewal process τ by taking the zero level set of S, i.e. $\tau_k = \inf\{i > \tau_{k-1} \;:\; S_i = 0\}$. On the other hand, given a renewal process we can always defined a random walk S such that $\{n \;:\; S_n = 0\} = \tau$, for instance we set $S_n = n - \sup\{k \;:\; \tau_k \le n\}$.

Let us observe that for the symmetric simple random walk Stirling's formula provides the asymptotics of $K(2n)$ (see e.g. [44, Appendix A.6]):

$$K(2n) = \mathrm{P}(\tau_1 = 2n) = \mathrm{P}\left(S_{2n} = 0, S_1 \neq 0, \cdots, S_{2n-1} \neq 0\right) \underset{n\to\infty}{\sim} \sqrt{\frac{1}{4\pi}}\frac{1}{n^{3/2}}.$$

In particular the state 0 is recurrent for the symmetric simple random walk. On the other hand, for a non-symmetric simple random walk, that is $\mathrm{P}(X_1 = 1) = p$, $\mathrm{P}(X_1 = -1) = 1 - p$, $p \neq 1/2$, the same kind of estimation provides $\mathrm{P}(\tau_1 < \infty) < 1$, which means that the random walk is transient. This motivates the fact that for a general renewal process $K(\cdot)$ is a probability on $\mathbb{N} \cup \{\infty\}$, with $K(\infty) := 1 - \sum_{n\in\mathbb{N}} K(n) \geq 0$. Whenever $K(\infty) > 0$ we say that the renewal process τ is *terminating*. This is equivalent to say that a.s. τ is given by a finite number of points. The class of renewal processes considered in this section generalizes the simple random walk case and it is indexed by an exponent $\alpha > 0$ which controls the polynomial behavior of $K(\cdot)$

$$K(n) \sim \frac{L(n)}{n^{1+\alpha}}, \quad n \to \infty, \tag{2.3}$$

where $L(\cdot)$ is a slowly varying function. We recall that $L : \mathbb{R}_+ \to \mathbb{R}_+$ is a slowly varying function if for any $x \in \mathbb{R}_+$ it holds that

$$\lim_{t\to\infty} \frac{L(tx)}{L(t)} = 1.$$

Important examples are provided by functions asymptotically equivalent to a constant — $L(t) \sim c_K$ — or to a logarithmic power — $L(t) \sim (\log t)^\gamma$, $\gamma \in \mathbb{R}$ — cf. [14].

Remark 2.1. Another interesting class of $K(\cdot)$ is provided by the stretched-exponential one, $K(n) \sim L(n)e^{-n^\gamma}$, as $n \to \infty$, $\gamma \in \mathbb{R}$, that recently has captured mathematical attention [62, 78] and it is the focus of the first result of this thesis, cf. Section 3.1.

To use the renewal process instead of the random walk in the definition of the pinning model (2.1) we replace the random walk S by the renewal process τ, obtaining

$$\mathrm{P}_{N,h}(\tau) = \frac{1}{\mathrm{Z}_h(N)}\left[\exp\left(h\sum_{n=1}^{N-1} \mathbb{1}_{n\in\tau}\right)\mathbb{1}_{N\in\tau}\right]\mathrm{P}(\tau), \tag{2.4}$$

where the notation $n \in \tau$ means that there exists $j \in \mathbb{N}$ such that $\tau_j = n$. For technical convenience we assume that

$$K(n) > 0, \quad \forall n \in \mathbb{N}, \tag{2.5}$$

so that $\mathrm{P}_{N,h}$ is well defined for any $N \in \mathbb{N}$. Any periodicity, that is $K(n) \neq 0$ only on $\ell\mathbb{N}$ for some $\ell > 1$, can be dealt with by restricting the definition (2.4) to $N \in \ell\mathbb{N}$.

Let us stress that in the homogeneous pinning model the partition function is completely explicit:

$$\mathrm{Z}_h(N) = \sum_{k=1}^{N} e^{hk}\mathrm{P}\left(\sum_{n=1}^{N} \mathbb{1}_{n\in\tau} = k\,,\, N \in \tau,\right) = \sum_{k=1}^{N} e^{hk} \sum_{\ell\in\mathbb{N}^k, |\ell|=N} \prod_{j=1}^{k} K(\ell_j),$$

where $|\ell| = \sum_{i=1}^{k} \ell_i$. Therefore

$$\mathrm{Z}_h(N) = \sum_{k=1}^{N} \sum_{\ell\in\mathbb{N}^k, |\ell|=N} \prod_{j=1}^{k} e^h K(\ell_j). \tag{2.6}$$

In the case in which $h < -\log(1 - K(\infty))$ this last expression says that

$$\mathrm{Z}_h(N) = \mathrm{P}\left(N \in \tau^{(h)}\right), \tag{2.7}$$

where $\tau^{(h)}$ is a terminating renewal process with $\mathrm{P}\left(\tau_1^{(h)} = n\right) = e^h K(n)$. On the other hand $h > -\log(1 - K(\infty))$ prevents $e^h K(n)$ to be a probability measure on $\mathbb{N} \cup \{\infty\}$, therefore we introduce a normalization factor, called *free energy* $F : \mathbb{R} \to \mathbb{R}_+$, defined as the unique solution of

$$\sum_n \exp(-F(h)n + h)K(n) = 1, \tag{2.8}$$

when such solution exists, which is the case $h \geq -\log(1 - K(\infty))$. Otherwise we define $F(h) := 0$ for any $h < -\log(1 - K(\infty))$. This choice is motivated by the fact that $F(-\log(1 - K(\infty))) = 0$.

Note that if the solution exists, then it is unique because $x \mapsto \sum_n \exp(-nx)K(n)$ is strictly monotonic. This implies that $F(h) > 0$ for any $h > -\log(1 - K(\infty))$. This allows to have a general form for the partition function:

$$\mathrm{Z}_h(N) = e^{NF(h)}\mathrm{P}\left(N \in \tau^{(h)}\right), \tag{2.9}$$

where $\tau^{(h)}$ is a renewal process with

$$\mathrm{P}\left(\tau_1^{(h)} = n\right) = \exp(-F(h)n + h)K(n). \tag{2.10}$$

Note that if $h < -\log(1 - K(\infty))$, then (2.9) is nothing but (2.7).

Summarizing the homogeneous pinning model $\mathrm{P}_{N,h}$ is still a law of a suitable renewal process conditioned to visit N and (2.4) can be written as follows: for any $0 \leq \ell_1 < \cdots < \ell_n = N$, $\ell_i \in \mathbb{N}$

$$\mathrm{P}_{N,h}(\tau_1 = \ell_1, \cdots, \tau_n = \ell_n) = \mathrm{P}\left(\tau_1^{(h)} = \ell_1, \cdots, \tau_n^{(h)} = \ell_n \middle| N \in \tau^{(h)}\right), \tag{2.11}$$

where $\tau^{(h)}$ is the renewal process of (2.9). Moreover there exists a critical value of h

$$h_c = -\log(1 - K(\infty)) \tag{2.12}$$

such that if $h < h_c$, then the renewal process τ is terminating and if $h > h_c$ is non-terminating. We can go beyond this statement, giving a quantitative estimation on the fraction of points of τ smaller than N, that is the quantity $\frac{1}{N}\sum_{n=1}^N \mathbb{1}_{n\in\tau}$. If $h > h_c$, then the law of τ_1 is $e^{-F(h)n+h}K(n)$, which has finite expectation, and the distance of two consecutive points is finite. Thus $\sharp\{\tau \cap [0, N]\} \approx m_h N$, for some constant $m_h > 0$. On the other hand if $h < h_c$, then the renewal process is terminating, thus with a finite number of points in $\mathbb{N}$. This suggests that $\sharp\{\tau \cap [0, N]\} = o(N)$, as $N \to \infty$. This results can be made rigorous: We observe that

$$\mathrm{E}_{N,h}\left[\frac{1}{N}\sum_{n=1}^N \mathbb{1}_{n\in\tau}\right] = \frac{\mathrm{d}}{\mathrm{d}h}\frac{1}{N}\log \mathrm{Z}_h(N) \tag{2.13}$$

and that

$$F(h) = \lim_{N\to\infty}\frac{1}{N}\log \mathrm{Z}_h(N). \tag{2.14}$$

Therefore a simple argument of convexity ensures that we can interchange the derivative with respect to h and the limit of $N \to \infty$, obtaining

$$F'(h) = \lim_{N\to\infty}\frac{\mathrm{d}}{\mathrm{d}h}\frac{1}{N}\log \mathrm{Z}_h(N) = \lim_{N\to\infty}\mathrm{E}_{N,h}\left[\frac{1}{N}\sum_{n=1}^N \mathbb{1}_{n\in\tau}\right]. \tag{2.15}$$

In particular $F'(h) > 0$ if $h > -\log(1 - K(\infty))$, and equal to 0 if $h < -\log(1 - K(\infty))$.

Remark 2.2. Let us stress that to prove (2.14), we consider $\mathrm{Z}_h(N)$ like in (2.9) and we show that for any $h \in \mathbb{R}$, the function $\mathrm{P}\left(N \in \tau^{(h)}\right)$ decreases polynomially as $N \to \infty$. This result follows by standard estimations on the renewal function provided by the renewal Theorem (2.82) and (2.85). For a detailed proof we refer to the one of [44, Proposition 1.1].

This result can be formulated as a convergence in probability of $\frac{1}{N}\sum_{n=1}^{N}\mathbb{1}_{n\in\tau}$ to $F'(h)$, cf. [45, Proposition 2.9]

Theorem 2.3. *For any $\varepsilon > 0$*

$$\lim_{N\to\infty} \mathrm{P}_{N,h}\left(\left|\frac{1}{N}\sum_{n=1}^{N}\mathbb{1}_{n\in\tau} - F'(h)\right| \le \varepsilon\right) = 1. \tag{2.16}$$

Moreover $F'(h) = 0$ if $h < -\log(1 - K(\infty))$, while, if $h > -\log(1 - K(\infty))$, $F'(h) > 0$.

Once we have identified the critical value h_c which defines the phase transition for the homogeneous pinning model, we can consider the interplay between such phase transition and the critical behavior of the free energy, that is the regularity of $F(h)$ when $h \downarrow h_c$. For this purpose we look for the asymptotic of $\Psi(x) = 1 - \sum_{n\in\mathbb{N}} e^{-xn}K(n)$ as $x \to 0$. Let us discuss this computation when $K(\cdot)$ is a probability measure on $\mathbb{N}$ satisfying (2.3), with $L(n)$ equal to a constant c_K, and (2.5). In this case $h_c = 0$ and

$$\begin{aligned}
\Psi(x) = 1 - \sum_{n=1}^{\infty} e^{-xn}K(n) &= 1 - \sum_{n=1}^{\infty} e^{-xn}\left(\sum_{j\ge n} K(j) - \sum_{j\ge n+1} K(j)\right) \\
=&\left(\sum_{j\ge 1} K(j) + \sum_{n=1}^{\infty} e^{-xn}\sum_{j\ge n+1} K(j)\right) - \sum_{n=1}^{\infty} e^{-xn}\sum_{j\ge n} K(j) \\
=&(1 - e^{-x})\sum_{n=0}^{\infty}\left(\sum_{j\ge n+1} K(j)\right)e^{-xn}.
\end{aligned}$$

If $\alpha > 1$ we conclude immediately that

$$\Psi(x) \underset{x\to 0}{\sim} xE[\tau_1], \tag{2.17}$$

while for $\alpha \in (0, 1)$ a Riemann sum approximation provides

$$\Psi(x) \underset{x\to 0}{\sim} x\sum_{n}\frac{c_K}{\alpha n^{\alpha}}e^{-xn} \underset{x\to 0}{\sim} \frac{c_K\, x^{\alpha}}{\alpha}\int_0^{\infty} s^{-\alpha}e^{-s}ds = c_K\frac{\Gamma(1-\alpha)}{\alpha}x^{\alpha}. \tag{2.18}$$

Recalling that $\Psi(F(h)) = 1 - e^{h}$, by inverting the asymptotics in (2.17) and (2.18), we obtain the critical behavior of $F(h)$. The precise and general statement is given by (cf. [44, Theorem 2.1])

Theorem 2.4. *For every choice of $\alpha \ge 0$ and $L(\cdot)$ in (2.3), there exists a slowly varying function $\hat{L}(\cdot)$ uniquely defined by $L(\cdot)$ such that*

$$F(h) \overset{h\searrow h_c}{\sim} (h - h_c)^{1/\min\{\alpha,1\}}\hat{L}(1/(h - h_c)). \tag{2.19}$$

In particular $\hat{L}(\cdot)$ is a constant if $\alpha > 1$.

As consequence of this theorem we have that if $\alpha > 1$, then $F(h)$ is not C^1 at the critical point and we say that the transition is of first order. In general we say that the model shows a phase transition of k-th order if $F(h)$ is C^{k-1} but not C^k at the critical point. The homogeneous pinning model has such a phase transition of k-th order, $k = 2, 3, \cdots$, at $h = h_c$ if $\alpha \in [1/k, 1/(k-1))$.

DNA denaturation: Poland - Scheraga model

We want conclude the section dedicated to the homogeneous model by explaining how it can be used to describe the DNA denaturation process. This goes under the name of Poland and Scheraga's model [71] and has captured the attention of several mathematicians, physicists and biologists, see [56, 65, 73] for a complete review on the subject. Here we are going to introduce the original model, without discussing how well it models the DNA denaturation's phenomenology. See [79] for this related topic.

The research on the DNA structure has a very long story starting at the end of 19-th century arriving in 1953 when James Watson and Francis Crick suggested what is now accepted as the first correct double-helix model of DNA structure [80]. Each helix is a long polymeric chain of monomeric units called nucleotids (a DNA molecule can have even several million nucleotides). Nucleotids are made up of three parts: a sugar, a phosphate group and a base. There are four types of (DNA) bases: adenine (A), cytosine (C), guanine (G) and thymine (T). The double-helix structure is realized via hydrogen bonds, which binds the two helix. A base does not couple with any other base: the base-pairing rules are A – T and G – C. Therefore the two helix constituting a double-helix DNA have to be complementary to realize the appropriate base-pairing. An interesting fact is that DNA double helix opens up for all sorts of reasons, including good ones, like transcription and replication that are at the basis of life. These phenomena are mediated by enzymes but it can be forced by external factor, like the temperature. It is known that there exists a critical temperature over which the DNA double-helix is broken and the two strands separate. When it happens we say that the DNA is denatured by the high temperature. In principle this critical temperature depends on the length of the DNA molecule and its specific nucleotids sequence.

The Poland and Scheraga's model proves an abstract DNA description which shows a phase transition. To introduce the model we idealize the DNA as a long alternating sequence of loops and bonded bases. We call $\mathtt{E} < 0$ the *binding energy* which force to bind together two bases-pairing and we can define $\mathcal{E}$ the *loop entropy* which defines the cost associated to have a loop of length ℓ. We assume

$$\mathcal{E}(\ell) \overset{\ell\to\infty}{\sim} \sigma\frac{\mu^\ell}{\ell^c}, \quad \text{and} \quad \mathcal{E}(\ell) > 0,\ \forall \ell > 0, \tag{2.20}$$

where $\mu > 1$ is a geometric factor, $c > 1$ is the *loop-closure* exponent and $\sigma > 0$ is the *cooperativity parameter*. These three parameters define the structure of the chain. As boundary condition we impose that for any DNA-chain of length N the first and the last pairs of nucleotids are bind together.

The probability of a given configuration with $n \leq N$ loops $\underline{\ell} = (\ell_1, \cdots, \ell_n)$ is equal to

$$\mathrm{P}_N^{\mathtt{PSM}}(\underline{\ell}) = \frac{1}{\mathrm{Z}_N^{\mathtt{PSM}}} e^{-\frac{\mathtt{E}}{T}n} \prod_{i=1}^{n} \mathcal{E}(\ell_i), \tag{2.21}$$

where $T > 0$ is the temperature of the system. We assume that $|\underline{\ell}| := \sum_{j=1}^n \ell_j = N$, the total length of the DNA molecule. In these notations $\ell_j = 1$ means that we have two consecutive bind bases-pairing. If we assume that $\mathcal{E}(\ell)/\mu^\ell$ is a probability on $\mathbb{N}$, then

$$\mathrm{Z}_N^{\mathtt{PSM}}\mu^{-N} = \sum_{n=1}^{N} \sum_{\substack{\underline{\ell} \\ |\underline{\ell}|=N}} \prod_{i=1}^{n} \exp\left\{-\frac{\mathtt{E}}{T}\right\} \frac{\mathcal{E}(\ell_i)}{\mu^\ell}. \tag{2.22}$$

In this formula we are averaging with respect to the law of the renewal process $K(\ell) = \frac{\mathcal{E}(\ell)}{\mu^\ell}$ and thus $\mathrm{Z}_N^{\mathtt{PSM}}\mu^{-N}$ is the partition function of an homogeneous pinning model with $h = -\frac{\mathtt{E}}{T}$. Therefore the free energy of the Poland and Scheraga model is given by

$$F^{\mathtt{PSM}}(T) = \log\mu + F(\mathtt{E}/T). \tag{2.23}$$

In such a way there exists a critical temperature $T_c > 0$ over which the presence of loops becomes predominant, leading to the denaturation of the DNA. The limitation of this model is that we do not take in account the sequence of different bases-pairing which form the DNA molecule. If we want to consider also this fact we can consider a pinning model in which the reward is not constant, but it changes site by site. This leads to the next section in which we are going to introduce such modification of the homogeneous pinning model.

2.3 Disordered pinning model

The *disordered pinning model* is defined as a *random* perturbation of the homogeneous one (2.4). For any i, $i = 1, \cdots, N$ we replace the exponent h by $\beta\omega_i + h$, where ω_i is a fixed (quenched) value, independent of the renewal process, which can change site by site. The sequence $\omega = (\omega_i)_{i\in\mathbb{N}}$ is called the disorder of the system. We generate the disorder by a *quenched* random sequence and we denote by $\mathbb{P}$ its law. Therefore for any $N \in \mathbb{N}$ we consider the family of probability $\mathrm{P}^{\omega}_{N,h,\beta}$, indexed by $h \in \mathbb{R}$, $\beta \in \mathbb{R}_+$ defined as

$$\mathrm{P}^{\omega}_{N,h,\beta}(\tau) = \frac{1}{\mathrm{Z}^{\omega}_{\beta,h}(N)} \left[\exp\left(\sum_{n=1}^{N} (\beta\omega_n + h)\mathbb{1}_{N\in\tau} \right) \mathbb{1}_{n\in\tau} \right] \mathrm{P}(\tau). \tag{2.24}$$

Since the realization of ω is fixed, we refer to (2.26) as the *quenched* model.

We assume that the random sequence ω is i.i.d. with some finite exponential moments. Precisely we assume that ω_1 has 0 mean and unit variance and there exists $\beta_0 > 0$ such that

$$\Lambda(\beta) = \log \mathbb{E}[e^{\beta\omega_1}] < \infty \quad \forall \beta \in (-\beta_0, \beta_0), \quad \mathbb{E}[\omega_1] = 0, \quad \mathbb{V}[\omega_1] = 1. \tag{2.25}$$

Under these assumptions it is useful to make a change of parametrization $h \mapsto h - \Lambda(\beta)$, in order to normalize the random variable $e^{\beta\omega_1}$ and we redefine (2.24) as

$$\mathrm{P}^{\omega}_{N,h,\beta}(\tau) = \frac{1}{\mathrm{Z}^{\omega}_{\beta,h}(N)} \left[\exp\left(\sum_{n=1}^{N} (\beta\omega_n - \Lambda(\beta) + h)\mathbb{1}_{n\in\tau} \right) \mathbb{1}_{n\in\tau} \right] \mathrm{P}(\tau). \tag{2.26}$$

Remark 2.5. The assumption on the exponential moments is very important for the results that we are going to present in the sequel, and different choices of disorder provide very different behavior of the pinning model. This is, for instance, the case of heavy-tailed random variables that we discuss in Section 3.1.

In analogy with the homogeneous model we study the behavior of the pinning model through the analysis of the free energy defined as

$$F(\beta, h) = \lim_{N\to\infty} \frac{1}{N} \mathbb{E}\left[\log \mathrm{Z}^{\omega}_{\beta,h}(N) \right], \tag{2.27}$$

where the partition function $\mathrm{Z}^{\omega}_{\beta,h}(N)$ it was introduced in (2.26). The existence of the free energy follows by a super-additive argument. The result can be upgraded by using Kingman's super-additive ergodic Theorem [58], which ensures that the r.h.s. converges a.s., that is

$$F(\beta, h) = \lim_{N\to\infty} \frac{1}{N} \log \mathrm{Z}^{\omega}_{\beta,h}(N), \quad \mathbb{P} - \text{a.s.}$$

See e.g. [44, Chapter 4].

Remark 2.6. The choice of $\mathbb{1}_{N\in\tau}$ in (2.26) is a boundary condition which constrains the polymer to go back to zero after N steps. Other interesting choices are provided by the free and the conditional one, for which the respective partition functions are defined as follows

$$Z^{\omega,\mathtt{f}}_{\beta,h}(N) = \mathrm{E}\left[\exp\left(\sum_{n=1}^{N}(\beta\omega_n - \Lambda(\beta) + h)\mathbb{1}_{n\in\tau}\right)\right], \tag{2.28}$$

$$Z^{\omega,\mathtt{c}}_{\beta,h}(M,N) = \mathrm{E}\left[\exp\left(\sum_{n=M+1}^{N-1}(\beta\omega_n - \Lambda(\beta) + h)\mathbb{1}_{n\in\tau}\right)\middle| M\in\tau,\, N\in\tau\right], \tag{2.29}$$

We stress that the free, the conditional and the original partition function differ at most by a polynomial factor, deriving from assumption (2.3). It follows that the free energy of such three partition functions is always the same as well as the critical properties.

To study the critical properties of the disordered pinning model we start to note that the homogeneous pinning models provides lower and upper bounds for the free energy:

$$F(0, h - \Lambda(\beta)) \le F(\beta, h) \le F(0, h). \tag{2.30}$$

In particular we obtain $F(\beta, h) \ge 0$, because the free energy of the homogeneous pinning model is non-negative. The right inequality, $F(\beta,h) \le F(0,h)$, is a straightforward consequence of Jensen's inequality and it goes under the name of *annealed* bound. Jensen's inequality provides the proof of the left inequality, but the computation is slightly less obvious:

$$\begin{aligned}\log Z_{\beta,h}(N) &= \log \mathrm{E}_{N,h-\Lambda(\beta)}\left[e^{\sum_{n=1}^{N}\beta\omega_n\mathbb{1}_{n\in\mathbb{N}}}\right] + \log Z_{N,h-\Lambda(\beta)}\\ &\ge \sum_{n=1}^{N}\beta\omega_n \mathrm{P}_{N,h-\Lambda(\beta)}(\tau_1 = N) + \log Z_{N,h-\Lambda(\beta)},\end{aligned}$$

where $\mathrm{E}_{N,h-\Lambda(\beta)}$ is the expectation of an homogeneous pinning model of parameter $h - \Lambda(\beta)$. By taking the expectation with respect to the disorder $\mathbb{E}$ we obtain the wished inequality, because $\mathbb{E}[\omega_n] = 0$.

Let us observe that for any fixed $\beta > 0$ the function $h \mapsto F(\beta, h)$ is monotonic and thus the *critical point*

$$h_c(\beta) := \sup\{h \,:\, F(\beta,h) = 0\} \tag{2.31}$$

is well defined and it separates the plane (h,β) in two regions $\mathcal{L} = \{(\beta,h) : F(\beta,h) > 0\}$ and $\mathcal{D} = \{(\beta,h) : F(\beta,h) = 0\}$. One can prove that an analogous version of Theorem 2.3 holds:

$$\begin{aligned}(\beta,h)\in\mathcal{L} \quad &\Leftrightarrow \quad \exists\, m_{\beta,h} > 0 : \forall\,\varepsilon > 0 \quad \mathrm{P}^{\omega}_{N,h,\beta}\left(\left|\frac{1}{N}\sum_{n=1}^{N}\mathbb{1}_{n\in\tau} - m_{\beta,h}\right| \le \varepsilon\right) \underset{N\to\infty}{\overset{\mathbb{P}-\text{probability}}{\longrightarrow}} 1,\\ (\beta,h)\in\mathring{\mathcal{D}} \quad &\Leftrightarrow \quad : \forall\,\varepsilon > 0 \quad \mathrm{P}^{\omega}_{N,h,\beta}\left(\left|\frac{1}{N}\sum_{n=1}^{N}\mathbb{1}_{n\in\tau}\right| \le \varepsilon\right) \underset{N\to\infty}{\overset{\mathbb{P}-\text{probability}}{\longrightarrow}} 1.\end{aligned} \tag{2.32}$$

See the monographs [44, 31, 45]. Moreover (2.30) gives

$$h_c(0) \le h_c(\beta) \le h_c(0) + \Lambda(\beta) \tag{2.33}$$

and $h_c(0)$ is the *annealed critical value*. We recall that $h_c(0) = -\log(1 - K(\infty))$. It is now natural to question if the presence of the disorder plays a role or not, see Figure 2.3. A heuristic but precise prediction is given by *the Harris criterion* [51].

2.3.1 The Harris criterion

The Harris criterion was proposed in the '70s to understand whether and how the introduction of a small amount of disorder in a homogeneous model should change its critical behavior. In the original paper [51] the model considered was the diluted Ising model, but the ideas developed are very useful to understand the presence of the disorder in many disordered models, among which we find the pinning one. In the pinning model this criterion has inspired different methods to understand when the disorder ω is irrelevant or not, namely when the critical properties of the quenched system differ from those of the annealed one [33, 40]. The anneal model is a particular homogeneous pinning model, cf. (2.34). What one means with critical properties is the behavior of the model near to criticality, i.e. when $h \approx h_c(\beta)$. In particular we aim to describe the way $F(\beta, h)$ vanishes as $h \searrow h_c(\beta)$, by comparing the critical exponent of the disordered model with the annealed one, which turns out to be the same of an homogeneous model (2.19). Whenever the critical exponent of the disordered model is different from the homogeneous one, even if β is arbitrarily small, we say that the disorder is relevant. In the pinning case the analysis goes beyond the critical exponents, by pushing such study to the structure of the critical point $h_c(\beta)$. The final answer is that the disorder is irrelevant if $\alpha < 1/2$ and relevant if $\alpha > 1/2$, where α is the exponent of the renewal process (2.3), see Figure 2.3. The case $\alpha = 1/2$ is more subtle and is called *marginal*: the relevance or irrelevance turns out to depend on the choice of the slowly varying function L in (2.3). Let us explain this fact through an heuristic argument inspired by the one used in [33, 40] and [45], which will be made rigorous in the sequel.

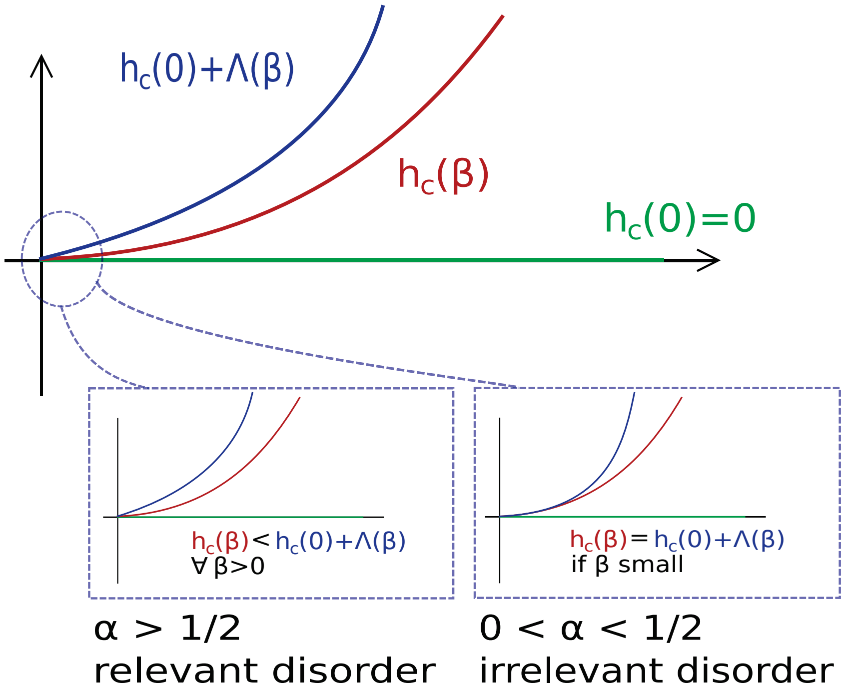

Fig. 2.3: The critical curve $h_c(\beta)$ is bounded by $h_c(0)$ from below and $h_c(0) + \Lambda(\beta)$ from above. Note that $\Lambda(\beta) \sim \beta^2/2 + o(\beta^2)$ as $\beta \to 0$.
Moreover the relevance / irrelevance of the disorder depends on the choice of α, the exponent of the renewal process: if $\alpha \in (0, 1/2)$, then the disorder is irrelevant and $h_c(\beta) = h_c(0) + \Lambda(\beta)$ if β is small enough, otherwise, if $\alpha > 1/2$ the disorder is relevant and $h_c(\beta) < h_c(0) + \Lambda(\beta)$.

Remark 2.7. Whenever we study the critical behavior of the model it is not restrictive to assume the renewal process to be non-terminating, indeed it is enough to operate a change of variables $h \mapsto h + h_c$ in the definition of the pinning model (2.26) and consider an alternative non-terminating renewal process τ' such that $K'(n) = K(n)/(1 - K(\infty))$. This is not true for the free model $Z^{\omega,\mathrm{f}}_{\beta,h}(N)$ in (2.28), but since the free energy and the critical exponents of such two models are the same, restricting our consideration on the non-terminating case is not restrictive even if we consider the free model. Without loss of generality, in the rest of this section we assume $K(\infty) = 0$ so that the annealed critical point $h_c(0) = 0$.

We start to fix a value of $\beta > 0$ and choose $h \geq 0$. We consider the *annealed* partition function

$$Z^{\mathrm{ann}}_h(N) := \mathbb{E}\left[Z^{\omega,\mathrm{f}}_{\beta,h}(N)\right] = \mathbb{E}\mathrm{E}\left[e^{\sum_{n=1}^N(\beta\omega_n - \Lambda(\beta)+h)\mathbb{1}_{n\in\tau}}\right] = \mathrm{E}\left[e^{h\sum_{n=1}^N \mathbb{1}_{n\in\tau}}\right], \tag{2.34}$$

which corresponds to the homogeneous model. We observe that

$$\frac{Z^{\omega,\mathrm{f}}_{\beta,h}(N)}{Z^{\mathrm{ann}}_h(N)} = \mathrm{E}^{\mathrm{ann}}_{h,N}\left[\exp\left\{\sum_{n=1}^N(\beta\omega_n - \Lambda(\beta))\mathbb{1}_{n\in\tau}\right\}\right], \tag{2.35}$$

where $\mathrm{E}^{\mathrm{ann}}_{h,N}$ is the expectation of the annealed (homogeneous) pinning model. We define

$$\zeta(x) = e^{\beta x - \Lambda(\beta)} - 1, \tag{2.36}$$

and we rewrite (2.35) as a polynomial

$$\begin{aligned}\mathrm{E}^{\mathrm{ann}}_{h,N}\left[\exp\left\{\sum_{n=1}^N(\beta\omega_n - \Lambda(\beta))\mathbb{1}_{n\in\tau}\right\}\right] &= \mathrm{E}^{\mathrm{ann}}_{h,N}\left[\prod_{n=1}^N(\zeta(\omega_n)\mathbb{1}_{n\in\tau} + 1)\right]\\ =&1 + \sum_{n=1}^N \zeta(\omega_n)\mathrm{P}^{\mathrm{ann}}_{h,N}(n\in\tau) + \cdots\end{aligned} \tag{2.37}$$

Note that $(\zeta(\omega_i))_{i\in\mathbb{N}}$ is an i.i.d. sequence of centered random variable with variance $\sim \beta^2$ as $\beta \to 0$, thus we approximate $\zeta(\omega_i)$ by $\beta\tilde{\omega}_i$, where $(\tilde{\omega}_i)_{i\in\mathbb{N}}$ is a sequence of standard Gaussian random variables. Therefore the polynomial expansion in (2.37) implies that

$$\mathbb{E}\log\frac{Z^{\omega,\mathrm{f}}_{\beta,h}(N)}{Z^{\mathrm{ann}}_h(N)} \approx -\frac{1}{2}\beta^2\sum_{n=1}^N \mathrm{P}^{\mathrm{ann}}_{h,N}(n\in\tau)^2 + \cdots \tag{2.38}$$

We observe that as long as $h > 0$ and n and $N - n$ go to ∞ as N grows to ∞ as well, then

$$\mathrm{P}^{\mathrm{ann}}_{h,N}(n\in\tau) = \frac{Z^{\mathrm{ann}}_h(n)Z^{\mathrm{ann}}_h(N-n)}{Z^{\mathrm{ann}}_h(N)} \overset{(2.9)}{=} \frac{\mathrm{P}\left(n\in\tilde{\tau}^{(h)}\right)\mathrm{P}\left(N-n\in\tilde{\tau}^{(h)}\right)}{\mathrm{P}\left(N\in\tilde{\tau}^{(h)}\right)} \underset{n,N-n\to\infty}{\sim} \frac{1}{\mathrm{E}\left[\tilde{\tau}^{(h)}\right]}$$

by the Renewal Theorem (2.82). Furthermore by (2.15) $\frac{1}{N}\sum_{n=1}^N \mathrm{P}^{\mathrm{ann}}_{h,N}(n\in\tau) \underset{N\to\infty}{\sim} F'(0,h)$, thus $F'(0,h) = 1/\mathrm{E}\left[\tilde{\tau}^{(h)}\right]$ by Cesàro's mean Theorem. These estimations suggest to replace $\mathrm{P}^{\mathrm{ann}}_{h,N}(n\in\tau)$ by $F'(0,h)$ in (2.38), obtaining

$$F(\beta,h) \approx F(0,h) - \frac{1}{2}\beta^2(F'(0,h))^2 + \cdots. \tag{2.39}$$

By (2.19) for $\alpha\in(0,1)$ we have $F(0,h)\approx h^{1/\alpha}$ and thus $F'(0,h)^2 \approx h^{2(1/\alpha-1)}$. In the case of $\alpha < 1/2$ we have that $h^{2(1/\alpha-1)}$ is negligible with respect to $h^{1/\alpha}$ and thus $F(\beta,h)\approx h^{1/\alpha}$ which is the annealed behavior. On the other hand if $\alpha > 1/2$, then the second term of the expansion is much larger of the first one, which is a symptom that something does not work in our expansion, e.g. we are expanding around the wrong point. To find the good one we note that $F(\beta, h) = 0$ for all $h \leq h_c(\beta)$, therefore $h_c(\beta)$ should be equivalent to the value where the right-hand side of (2.39) vanishes. This means that $h_c(\beta) > 0\,(= h_c(0))$ and

$$h_c(\beta) \approx \beta^{\frac{2\alpha}{2\alpha-1}}, \qquad \text{for } \beta \text{ small.} \tag{2.40}$$

Most of these results have been made rigorous. In particular the irrelevance of the disorder when $\alpha < 1/2$ has been proven in [4, 26, 60] and in [49] it has been shown that the critical exponent of the disordered model must be bigger than 2. This means that if $\alpha > 1/2$, then it is strictly bigger than the annealed critical exponent, which is equal to $\frac{1}{\alpha}$ cf. (2.19).

The case $\alpha \in (1/2, 1)$ has been investigated in depth [3, 32], confirming (2.40) only up to non-matching constants: there is a slowly varying function $\tilde{L}_\alpha$ (determined explicitly by L and α, see Remark 5.2 below), and constant $0 < c < \infty$ such that for $\beta > 0$ small enough

$$c^{-1}\tilde{L}_\alpha(\beta^{-1})\beta^{\frac{2\alpha}{2\alpha-1}} \leq h_c(\beta) \leq c\tilde{L}_\alpha(\beta^{-1})\beta^{\frac{2\alpha}{2\alpha-1}}. \tag{2.41}$$

The marginal case (in which the symmetric simple random walk falls), $\alpha = 1/2$, has been considered in several work along the last years [4, 46, 47, 77], and it has solved only recently by Berger and Lacoin [10]. In such case the critical properties of the model depend on the slowly varying function choice L and $h_c(\beta) > 0$ for any β (and thus it differs from the annealed one) if and only if $\sum_n \frac{1}{n\,L(n)^2} = \infty$. In [9] it was considered the case $\alpha > 1$, and the precise asymptotics of $h_c(\beta)$ as $\beta \to 0$ was found:

$$h_c(\beta) \underset{\beta\to 0}{\sim} \frac{\alpha}{2\mathrm{E}\,[\tau_1]\,(1+\alpha)}\beta^2 \tag{2.42}$$

In this thesis — Section 3.2 — we find the exact asymptotics of $h_c(\beta)$ as $\beta \to 0$ when $\alpha \in (1/2, 1)$, by sharpening (2.41) and making rigorous the deduction of (2.40). In particular we prove that

$$h_c(\beta) \underset{\beta\to 0}{\sim} m_\alpha \tilde{L}_\alpha(\beta^{-1})\beta^{\frac{2\alpha}{2\alpha-1}} \tag{2.43}$$

where m_α is an universal constant depending only on α and it is given by the critical point of a suitable related continuum model [23, 24], which we are going to introduce and discuss in the next section.

2.3.2 The weak disorder regime

The aim of this section is to introduce and define the *continuum partition function*, showing how it appears naturally as continuum limit of the (discrete) partition function of the pinning model. The homogeneous case has been consider by Sohier [76], while the disordered case by Caravenna, Sun and Zygouras [23, 24].

We assume the disorder to be an i.i.d. random sequence $(\omega = (\omega_i)_{i\in\mathbb{N}}, \mathbb{P})$ satisfying (2.25), while the renewal process $(\tau = (\tau_i)_{i\in\mathbb{N}}, \mathrm{P})$ is non-terminating and it satisfies (2.3) with $\alpha \in (1/2, 1)$, and (2.5).

In analogy with (2.36), we define

$$\zeta(x) = e^{\beta x - \Lambda(\beta) + h} - 1$$

and by observing that

$$e^{(\beta x - \Lambda(\beta) + h)\mathbb{1}_{n\in\tau}} = 1 + \left(e^{\beta x - \Lambda(\beta) + h} - 1\right)\mathbb{1}_{n\in\tau},$$

the *free* partition function (2.28) admits a polynomial expansion

$$\begin{aligned} \mathrm{Z}^{\omega,\mathfrak{f}}_{\beta,h}(N) =& \mathrm{E}\left[1+\sum_{k=1}^{N}\sum_{\underline{n}\in\{1,\cdots,N\}^k_{\leq}}\zeta(\omega_{n_i})\cdots\zeta(\omega_{n_k})\mathbb{1}_{n_1\in\tau}\cdots\mathbb{1}_{n_k\in\tau}\right]= \\ =&1+\sum_{k=1}^{N}\sum_{\underline{n}\in\{1,\cdots,N\}^k_{\leq}}\zeta(\omega_{n_i})\cdots\zeta(\omega_{n_k})\mathrm{P}\left(n_1\in\tau,\cdots,n_k\in\tau\right), \end{aligned} \tag{2.44}$$

where $\{1,\cdots,N\}^k_{\leq} := \{(n_1,\cdots,n_k)\mid 1\leq n_1\leq\cdots\leq n_k\leq N\}$ is the set of the ordered sequence of length k taking values in $\{1,\cdots,N\}$. Note that this is the same polynomial expansion made in (2.37), but here we consider all the terms in the sum. Let us explain heuristically how to find the good rescaling for h and β and why this polynomial expansion converges. Below we give the precise statement. To simplify some technical step we assume that $K(n)\sim c/n^{1+\alpha}$, with $c>0$ a fixed constant. We observe that $(\zeta(\omega_n))_{n\in\mathbb{N}}$ is an i.i.d. sequence of expectation $\mathbb{E}\left[\zeta(\omega_1)\right]=e^h-1\sim h$ and variance $\mathbb{V}\left[\zeta(\omega_1)\right]=e^{2h}\left(e^{2\Lambda(\beta)}-1\right)\sim\Lambda(2\beta)-2\Lambda(\beta)\sim\beta^2$ as $h,\beta\to 0$, therefore $\zeta(\omega_n)$ can be approximated by Gaussian random variable, namely

$$\zeta(\omega_n)\approx\beta\tilde{\omega}_n+h$$

where $(\tilde{\omega}_n)_{n\in\mathbb{N}}$ is a sequence of i.i.d. standard Gaussian random variables. Let us consider the first term, $k=1$, in (2.44). Its law is approximated by a Gaussian random variable

$$\sum_{n=1}^{N}\zeta(\omega_n)\mathrm{P}\left(n\in\tau\right)\approx\sum_{n=1}^{N}(\beta\tilde{\omega}_n+h)\mathrm{P}\left(n\in\tau\right)\overset{(d)}{=}\mathcal{N}\left(h\sum_{n=1}^{N}\mathrm{P}\left(n\in\tau\right),\beta^2\sum_{n=1}^{N}\mathrm{P}\left(n\in\tau\right)^2\right).$$

By (2.87) we have $\mathrm{P}\left(n\in\tau\right)\sim C/n^{1-\alpha}$, where $C>0$ is an explicit constant depending on $c>0$. Thus

$$\sum_{n=1}^{N}\zeta(\omega_n)\mathrm{P}\left(n\in\tau\right)\approx\mathcal{N}\left(h\,CN^{\alpha},\beta^2\,CN^{2\alpha-1}\right).$$

Therefore to have something of non-trivial in the limit $N\to\infty$ we have to rescale β,h as

$$\beta_N\propto\hat{\beta}N^{\frac{1}{2}-\alpha},\quad h_N\propto\hat{h}N^{-\alpha},\quad\text{for fixed }\hat{\beta}\in\mathbb{R}_+,\ \hat{h}\in\mathbb{R}. \tag{2.45}$$

By using this scaling and replacing $\tilde{\omega}_n$ by W_n-W_{n-1}, with $(W_t)_{t\geq 0}$ a standard Brownian motion, we obtain

$$\begin{aligned} \sum_{n=1}^{N}\zeta(\omega_n)\mathrm{P}\left(n\in\tau\right)\approx&\sum_{n=1}^{N}(\beta_N\tilde{\omega}_n+h_N)\frac{C}{n^{1-\alpha}} \\ \overset{(d)}{=}&\underbrace{\frac{\hat{h}}{N}\sum_{t\in\frac{\mathbb{N}}{N}\cap[0,1]}\frac{c_1}{t^{1-\alpha}}}_{\text{deterministic Riemann sum}}+\underbrace{\hat{\beta}\sum_{t\in\frac{\mathbb{N}}{N}\cap[0,1]}\frac{c_1}{t^{1-\alpha}}(W_t-W_{t-\frac{1}{N}})}_{\text{discrete Ito's integral}} \\ \overset{(d)}{\to}&\int_0^1\frac{c_1}{t^{1-\alpha}}(\hat{h}\,dt+\hat{\beta}\,W(dt)),\ N\to\infty, \end{aligned}$$

where $c_1>0$ is a suitable constant depending on c,C. It is remarkable that the rescaling of β and h in (2.45) guarantees also the convergence of the other terms in the sum (2.44) and the limit is given by a sum of multiple stochastic integrals, called Weiner chaos expansion, and it defines $\mathbf{Z}^W_{\hat{\beta},\hat{h}}(1)$, the continuum partition function,

$$\mathbf{Z}^{W}_{\hat\beta,\hat h}(1) := 1 + \sum_{k=1}^{\infty} \int \cdots \int_{0\le t_{\le}\cdots \le t_k \le 1} \psi_{k,t}(t_1, \cdots, t_k) \prod_{i=1}^{k} (\hat\beta W(dt_i) + \hat h dt_i), \tag{2.46}$$

where $W = (W_t)_{t\ge 0}$ is a standard Brownian Motion and $\psi_{k,t}$ is an explicit symmetric deterministic function, called kernel. Note that the integrals in (2.46) can be viewed as ordinary Ito's integrals: it suffices to first integrate over $(\hat\beta W(dt_1) + \hat h dt_1)$ for $t_1 \in (0, t_2)$, then over $(\hat\beta W(dt_2) + \hat h dt_2)$ for $t_2 \in (0, t_3)$ and so on. For more details about the multiple stochastic integration we refer to [54].

Let us give the precise result: consider the processes $\left(\mathrm{Z}^{\omega,\mathtt{f}}_{\beta,h}(Nt)\right)_{t\ge 0}$ and $\left(\mathrm{Z}^{\omega,\mathtt{c}}_{\beta,h}(Ns, Nt)\right)_{t>s\ge 0}$ defined respectively in (2.28) and (2.29) if $Nt, Ns \in \mathbb{N}$ and then extended linearly to all possible positive real values. We have defined a sequence of random variables in the space of the continuous functions $C([0,\infty))$ and $C([0,\infty)^2_{\le})$ respectively. Here we recall the notation

$$[0,\infty)^2_{\le} := \{(u, v) \in \mathbb{R}_+ \times \mathbb{R}_+ \,:\, u \le v\}.$$

We equip these spaces with the local uniform topology: a sequence converges in this topology if and only if it converges uniformly on each compact set. Then cf. [24, Theorem 2.1 and Remark 2.3]

Theorem 2.8. *Let $(\omega = (\omega_i)_{i\in\mathbb{N}}, \mathbb{P})$ be an i.i.d. random sequence satisfying* (2.25) *and let $(\tau = (\tau_i)_{i\in\mathbb{N}}, \mathrm{P})$ be a non-terminating renewal process satisfying* (2.3) *for some $\alpha \in (1/2, 1)$,* (2.5) *and the following technical assumption on the* renewal function $u(n) := \mathrm{P}(n \in \tau)$*:*

$$\exists\, C, n_0 \in (0,\infty);\ \varepsilon, \delta \in (0,1] : \quad \left|\frac{u(\ell+n)}{u(n)} - 1\right| \le C\left(\frac{\ell}{n}\right), \quad \forall n \ge n_0,\ 0 \le \ell \le \varepsilon n. \tag{2.47}$$

For $N \in \mathbb{N}$, $\hat\beta \in \mathbb{R}_+$ and $\hat h \in \mathbb{R}$ we define

$$\beta_N = \begin{cases} \hat\beta \frac{L(N)}{N^{\alpha-\frac12}},\ if\ \alpha \in (\frac12, 1), \\ \frac{\hat\beta}{\sqrt{N}},\ if\ \alpha \in (1,\infty); \end{cases} \quad , \quad h_N = \begin{cases} \hat h \frac{L(N)}{N^{\alpha}},\ if\ \alpha \in (\frac12, 1), \\ \frac{\hat h}{N},\ if\ \alpha \in (1,\infty). \end{cases} \tag{2.48}$$

Then the sequence of processes $\left(\mathrm{Z}^{\omega}_{\beta_N,h_N}(Nt)\right)_{t\ge 0, N\in\mathbb{N}}$ and $\left(\mathrm{Z}^{\omega,\mathtt{c}}_{\beta_N,h_N}(Ns, Nt)\right)_{t>s\ge 0, N\in\mathbb{N}}$ converge in distribution in the space of continuous functions $C([0,\infty))$ and $C([0,\infty)^2_{\le})$ respectively. The two limit processes $\left(\mathbf{Z}^{W}_{\hat\beta,\hat h}(t)\right)_{t\ge 0}$ and $\left(\mathbf{Z}^{W,\mathtt{c}}_{\hat\beta,\hat h}(t)\right)_{t\ge 0}$, called continuum free partition function and continuum conditional partition function*, are defined by a Wiener chaos expansion as in* (2.46)*. The kernel of such expansions is*

$$\psi^{\mathtt{f}}_{k,t}(t_1, \cdots, t_k) = \begin{cases} \frac{C_\alpha^k}{t_1^{1-\alpha}} \prod_{i=1}^{k-1} \left(\frac{1}{(t_i - t_{i-1})^{1-\alpha}}\right),\ if\ \alpha \in (\frac12, 1), \\ \frac{1}{\mathrm{E}[\tau_1]^k},\ if\ \alpha \in (1,\infty); \end{cases} \tag{2.49}$$

$$\psi^{\mathtt{c}}_{k,s,t}(t_1, \cdots, t_k) = \begin{cases} \frac{C_\alpha^k}{(t_1 - s)^{1-\alpha}} \prod_{i=1}^{k-1} \left(\frac{1}{(t_i - t_{i-1})^{1-\alpha}}\right) \frac{(t-s)^{1-\alpha}}{(t-t_k)^{1-\alpha}},\ if\ \alpha \in (\frac12, 1), \\ \frac{1}{\mathrm{E}[\tau_1]^k},\ if\ \alpha \in (1,\infty), \end{cases} \tag{2.50}$$

where $C_\alpha = \frac{\alpha \sin \pi\alpha}{\pi}$ and the superscript $\mathtt{f}$ and $\mathtt{c}$ are referred respectively to the free and conditioned model.

Remark 2.9. Let us stress that (2.47) is a mild assumption: it was shown by Alexander [5] that for any $\alpha \in (0,1)$ and slowly varying function L there exists a Markov chain on $\mathbb{N}_0$ with ± 1 increments, called Bessel-like random walk, such that its return time to 0, denoted by σ, satisfies

$$\mathrm{P}(\sigma = 2n) \sim \frac{L(n)}{n^{1+\alpha}}, \quad \text{as } n \to \infty$$

and (2.47) holds for any $\delta < \alpha$, see [24, Lemma B.2].

2.3.3 Universality in the weak coupling limit

In the previous section we have introduced the continuum disordered pinning model. We have shown that when the disorder is relevant, i.e. $\alpha > \frac{1}{2}$ cf. Section 2.3.1, it approximates the disordered pinning model in the weak disorder regime. We are interested in knowing if such model provides an approximation of the critical behavior of the disordered pinning model, that is if the free energy and the critical point, suitably rescaled, converge to the analogous quantities described by the continuum model. Such challenging problem has been already considered (and solved) for the Copolymer model [22, 16], a relative model of the pinning one. Also for the pinning model the answer is not taken for granted, as showed by the case $\alpha > 1$: in this case it turns out that the continuum partition function has an explicit expression, cf. [23]

$$\mathbf{Z}^W_{\hat\beta,\hat h}(t) \overset{(d)}{=} \exp\left\{ \frac{\hat\beta}{\mathrm{E}[\tau_1]} W_t + \left(\frac{\hat h}{\mathrm{E}[\tau_1]} - \frac{\hat\beta^2}{2\mathrm{E}[\tau_1]^2} \right) t \right\},$$

which provides the exact value of the *continuum free energy*

$$\mathbf{F}^\alpha(\hat\beta, \hat h) = \lim_{t\to\infty} \mathbb{E}\left[\frac{1}{t} \log \mathbf{Z}^W_{\hat\beta,\hat h}(t) \right] = \frac{\hat h}{\mathrm{E}[\tau_1]} - \frac{\hat\beta^2}{2\mathrm{E}[\tau_1]^2}.$$

Such expression can take negative values for suitable choice of $\hat h$ and $\hat\beta$. This means that for $\alpha > 1$ the continuum free energy $\mathbf{F}^\alpha(\hat\beta, \hat h)$ does not capture the asymptotic behavior of its discrete counterparts $\mathrm{F}(\beta, h)$ in the weak coupling regime $h, \beta \to 0$. In particular the formal interchange of the limits $t \to \infty$ and $N \to \infty$

$$\begin{aligned} \mathbf{F}^\alpha(\hat\beta, \hat h) &= \lim_{t\to\infty} \frac{1}{t} \mathbb{E}\left[\log \mathbf{Z}^W_{\hat\beta,\hat h}(t) \right] = \lim_{t\to\infty} \frac{1}{t} \lim_{N\to\infty} \mathbb{E}\left[\log \mathrm{Z}^\omega_{\beta_N,h_N}(Nt) \right] \\ &= \lim_{N\to\infty} N \lim_{t\to\infty} \frac{1}{Nt} \mathbb{E}\left[\log \mathrm{Z}^\omega_{\beta_N,h_N}(Nt) \right] = \lim_{N\to\infty} N\, F(\beta_N,\, h_N). \end{aligned} \tag{2.51}$$

is not justified, precisely it fails in such situation.

The interchanging of such limits in (2.51) is in general a delicate issue. In the copolymer model it has been shown that (2.51) fails for $\alpha > 1$, but it holds for $\alpha < 1$ cf. [16, 22]. Motivated by such analogy with the copolymer model, we expect that for the pinning model (2.51) holds when $\alpha \in (\frac{1}{2}, 1)$. Moreover in such case it has been proven in [24] that the continuum partition function allows to define the *continuum pinning model*, which turns out to be the limit in distribution of the pinning model [24, Theorem 1.3]. We discuss briefly its construction in Section 2.5.5. Summarizing it is natural to conjecture that for $\alpha \in (1/2, 1)$ the continuum pinning model captures the critical behavior of the discrete pinning model. For this purpose we define the *continuum critical point* as

$$\mathbf{h}^\alpha_c(\hat\beta) = \inf\left\{ \hat h > 0 \,:\, \mathbf{F}^\alpha(\hat\beta, \hat h) > 0 \right\}. \tag{2.52}$$

By (3.23) it holds that

$$\mathbf{h}^\alpha_c(\hat\beta) = \hat\beta^{\frac{2\alpha}{2\alpha-1}} \mathbf{h}^\alpha_c(1).$$

On the other hand let us recall the known bounds on the discrete critical point $h_c(\beta)$ (2.41): for any

$\alpha \in (\frac{1}{2}, 1)$ there exists $c > 0$ and β_0 such that

$$c^{-1}\beta^{\frac{2\alpha}{2\alpha-1}}\tilde{L}_\alpha(\beta^{-1}) \le h_c(\beta) \le c\beta^{\frac{2\alpha}{2\alpha-1}}\tilde{L}_\alpha(\beta^{-1}), \quad \forall \beta < \beta_0, \tag{2.53}$$

where $\tilde{L}_\alpha(\cdot)$ is a slowly varying function uniquely defined by α and $L(\cdot)$ in (2.3). It is therefore natural to conjecture that

$$\lim_{\beta\to 0} \frac{h_c(\beta)}{\beta^{\frac{2\alpha}{2\alpha-1}}\tilde{L}_\alpha(\beta^{-1})} = \mathbf{h}_c(1). \tag{2.54}$$

The proof of such conjecture is one of the main results of this thesis and we prove it in Section 3.2. Let us stress the *universality* value of (2.54): the asymptotic behavior of $h_c(\beta)$ as $\beta \to 0$ depends only on the *tail* of the return time distribution $K(n) = \mathrm{P}(\tau_1 = n)$, through the exponent $\alpha \in (\frac{1}{2}, 1)$ and the slowly varying function L (which determine $\tilde{L}_\alpha$): all finer details of $K(n)$ beyond these key features disappear in the weak disorder regime. The same holds for the disorder variables: any admissible distribution for ω_1, cf. (3.18)-(3.19) below, has the same effect on the asymptotic behavior of $h_c(\beta)$.

2.4 A related model: Directed polymers in a random environment

In this section we are going to present the directed polymer in a random environment. Such model owns several connections with the pinning model. In this thesis we highlight the heavy tail case: in one hand to approach the analysis of the pinning model with heavy-tailed disorder, cf. Section 3.1, we use the ideas for the analogous case of directed polymer in a random environment with heavy-tails. On the other hand as byproduct of our techniques developed to study the pinning model with heavy-tailed disorder, we improve a result for the directed polymer in a random environment with heavy-tails.

Originally introduced in [52], the *directed polymer in a random environment* is a model to describe an interaction between a polymer chain and a medium with microscopic impurities (the external environment). In this model the polymer interacts with any part of the environment. Mathematically this means that any time at which the random walk visits a different state, its law is perturbed by a reward/penalty. Such rewards/penalties are time depending, i.e. each time the simple random walk visits a given state, the reward/penalty received changes. In the pinning model this interaction can happen only with a given single state. As in the pinning model, the environment choice can change substantially the behavior of the polymer: in [28, 29] one can find general and detailed reviews of the model in the most studied case in which the environment has finite exponential moments, while in [7] the case with heavy tails is considered. As well as the pinning model, also the directed polymer in random environment admits a continuum limit, the *continuum directed polymer in a continuum random environment* [1, 2, 23].

2.4.1 The model

Let $((S = (S_n)_{n\in\mathbb{N}}, \mathrm{P})$ be a d - dimensional simple random walk starting from 0. This means that $(S_n - S_{n-1})_{n\in\mathbb{N}}$ is an i.i.d. sequence such that $\mathrm{P}(S_n - S_{n-1} = e_i) = 1/(2d)$, where e_i is the i-th vector of the canonical base of $\mathbb{Z}^d$. In the following we perturb S up to time N, then it is useful to consider $\mu_N(\cdot)$, the law of the first N-steps of the simple random walk,

$$\mu_N(s_1, \cdots, s_N) = \mathrm{P}(S_1 = s_1, \cdots, S_N = s_N) = (2d)^{-N}.$$

The random environment is a random sequence $\left(\omega = (\omega_{i,j})_{i\in\mathbb{N}, j\in\mathbb{Z}^d}, \mathbb{P}\right)$ placed on all possible points of $\mathbb{N}\times\mathbb{Z}^d$.

By using these two ingredients, we fix a typical realisation of the environment and for a given simple random walk path of length N, $s = (s_1, \cdots, s_N) \subset \mathbb{Z}^d$, we define the *polymer measure*

$$\mu_{\beta,N}(s) = \frac{e^{\beta\sigma_N(s)}}{Z_{\beta,N}}\mu_N(s). \tag{2.55}$$

$\sigma_N(\cdot) = \sum_{i=1}^N \omega_{i,s_i}$ is called *Hamiltonian function* and it describes the energy due to the interaction of the random walk with the different states. Let us stress that it depends on a quenched realization of the environment. $Z^{\omega}_{\beta,N}$ is a normalization constant called partition function

$$Z_{\beta,N} = \sum_{\substack{s=(s_1,\cdots,s_N)\subset\mathbb{Z}^d:\\ \|s_i-s_{i-1}\|=1}} \exp\{\beta\sigma_N(s)\}\,\mu_N(s). \tag{2.56}$$

Note that if $\beta = 0$, then $Z^{\omega}_{\beta,N} = 1$ and we do not have any interaction with the environment and we recover the original law μ_N.

The main question is the same as for the pinning model: how does the environment affect the spatial configuration of the polymer chain as function of β and d, while N gets large? Analogously to the pinning model, the understanding of the problem comes from an energy-entropy argument. The energy of a trajectory is given by $\sigma_N(s)$, while the entropy cost is connected to the probability to have such trajectory. The interplay between the energy and the entropy is tuned by the coefficient β. For $\beta = 0$ the polymer measure is the law of the simple random walk, which shows a diffusive behavior, and if β is very small, then it is possible that the behavior of the simple random walk is not significantly perturbed by the presence of the environment. In such case we say that there is an entropy domination. For β large any entropic cost should be overcome by visiting a trajectory with positive energy. This suggests that in this case we may have the existence of distinguished paths with high energy around which the typical trajectory of the polymer concentrates, and superdiffusive phenomena can appear. In the extreme case of $\beta = \infty$ we do not have any entropic cost and the problem is to study the trajectories which maximize the energy. Such problem goes under the name of *last-passage percolation* [66]. In such case we say that there is an energy domination. Entropy domination is called *weak disorder*, and energy domination is called *strong disorder*. We expect to have a critical point β_c which separates the weak disorder from the strong one.

2.4.2 Directed polymers in a random environment with finite exponential moments

The most well studied case is the one where the environment $\left((\omega_{i,j})_{i\in\mathbb{N},j\in\mathbb{Z}^d}, \mathbb{P}\right)$ is an i.i.d. sequence with locally finite exponential moments, precisely

$$\mathbb{E}\left[\omega_{i,j}\right] = 0, \quad \mathbb{V}\left[\omega_{i,j}\right] = 1, \quad \Lambda(\beta) := \log \mathbb{E}\left[e^{\beta\omega_{i,j}}\right] < \infty, \forall\beta \in (-\beta_0, \beta_0), \tag{2.57}$$

for some $\beta_0 > 0$. Since our main focus will be an environment with heavy tails — presented in the next section — in this section we recall only some of the many known results, for a more complete review on such topic we refer to [28, 29].

In this case the precise separation between the weak and strong disorder is defined in terms of the positivity of the limit of the martingale $e^{-\Lambda(\beta)N} Z_{\beta,N}$: the weak disorder is given by the set

$$\left\{\beta \geq 0 \;:\; \lim_{N\to\infty} e^{-\Lambda(\beta)N} Z_{\beta,N} > 0\right\}, \tag{2.58}$$

while its complementary is the strong disorder. Note that $e^{-\Lambda(\beta)N} Z_{\beta,N}$ is a martingale with respect to the filtration of the environment viewed by the path $\mathcal{F}_n = \sigma(\omega_{i,j} \;:\; j \leq n,\, i \in \mathbb{Z}^d)$. In [29] the authors showed that in any dimension there exists a critical value of β, denoted by β_c, which divides the weak disorder from the strong one. If $d = 1, 2$, then $\beta_c = 0$, so that all $\beta > 0$ are in the strong disorder regime, while if $d \geq 3$, then $\beta_c > 0$. Understanding the polymer behavior in the strong disorder regime is a very challenging problem. At least in low dimensions one of the most well-studied phenomena is the *superdiffusivity* of the paths under the polymer measure. Superdiffusivity means that there exists an exponent $\zeta > 1/2$ which draws the fluctuation of the random walk, that is

$$S_n \sim |n|^{\zeta}, \quad n \to \infty,$$

for any typical realization of ω. It is conjectured that in $d = 1$

$$\zeta = \frac{2}{3}, \tag{2.59}$$

and it independent of the disorder. Such conjecture has recently been obtained in special models [8, 74] and in [15] it is claimed that such exponent should be valid as long as environment possesses more than five moments.

2.4.3 Directed polymers in a random environment with heavy tails

The results that we are going to present can be generalized to any dimension, but for sake of simplicity we present the problem in dimension 1 + 1, which is the dimension studied in depth in [7, 50].

Let us consider as environment an i.i.d. sequence of random variables with heavy tails, namely

$$\exists\, \alpha \in (0,2) \ : \quad \mathbb{P}(\omega_{i,j} > t) \sim L_0(t) t^{-\alpha}, \quad t \to \infty \tag{2.60}$$

where $L_0(\cdot)$ is a slowly varying function, cf. [14]. This assumption can be formulated in terms of *distribution function* of ω, $F(t) = \mathbb{P}(\omega_{1,1} \leq t)$, by requiring that $1 - F$ be α - regularly varying, with index $\alpha \in (0, 2)$, i.e.,

$$\forall\, s \in (0,\infty) \quad \lim_{t\to\infty} \frac{1 - F(ts)}{1 - F(t)} = s^{-\alpha}. \tag{2.61}$$

Moreover, mainly for technical reasons, we assume that the law of $\omega_{1,1}$ has no atom and it is supported in $(0, \infty)$. The reference example to consider is given by the Pareto Distribution.

In the heavy tails case the martingale approach discussed above does not work ($\Lambda(\beta) = \infty$) and the strategy starts by considering a simple random walk constrained to be 0 after N-steps and the *ordered statistics* of the environment that can be touched by such random walk. In the next section we discuss the last passage percolation with heavy tails [50], by introducing the ordered statistics of the environment.

Ordered statistics

Let us consider the *last-passage percolation with heavy tails* studied in [50]. This problem correspond to the case $\beta = \infty$ in the model of directed polymer in a random environment.

In the last-passage percolation with heavy-tailed disorder we consider the box $\Lambda_N = \{1, \cdots, N\}^2 \subset \mathbb{N}^2$, and at each point $x = (x_1, x_2)$ of such box we associate a *positive* weight ω_x. The aim is to study the following two quantities

$$\begin{aligned} \text{the } \textit{weight function} \quad \tilde{T}^{(N)} &= \max_{\pi\in\Pi^{(N)}} \sum_{v\in\pi} \omega_v, \\ \pi^{(N)} &= \arg\max_{\pi\in\Pi^{(N)}} \sum_{v\in\pi} \omega_v, \end{aligned} \tag{2.62}$$

where $\Pi^{(N)}$ is the set of all directed paths π between $(1, 1)$ and (N, N). We recall that a directed path $\pi = (x^0, \cdots, x^N)$ is an *ordered trajectory*, i.e. $x^k \leq x^{k+1}$, such that $\|x^{k+1} - x^k\| = 1$. We recall that $x \leq x'$

means that $x_i \le x_i'$, for any $i = 1, 2$

We generate the weights randomly, i.e., $\left(\omega = (\omega_x)_{x\in\Lambda_N}, \mathbb{P}\right)$ are i.i.d. strictly positive random variables whose tail is regularly varying with index $\alpha \in (0, 2)$, cf. (2.61) and we assume that ω_x has no atom. Note that absence of atoms ensures that a.s. $\pi^{(N)}$ is unique. Let us reformulate (2.62) in terms of the *ordered statistics* $\tilde{M}_1^{(N)} > \tilde{M}_2^{(N)} > \cdots > \tilde{M}_{N^d}^{(N)}$, where $\tilde{M}_1^{(N)}$ is the maximum value of $\omega = (\omega_x)_{x\in\Lambda_N}$, $\tilde{M}_2^{(N)}$ the second one and so on. By using the i.i.d. structure of ω, we can rewrite the disorder as

$$(\omega_x)_{x\in\Lambda_N} \overset{(d)}{=} \left(\tilde{M}^{(N)}_{\tilde{Y}_i^{(N)}}\right)_{i=1}^{N^2}, \tag{2.63}$$

where $(\tilde{Y}_i^{(N)})_{i\in\Lambda_N}$ is a random permutation of the points of Λ_N.

Extreme value theory, [35, 72], tells us that there exists a parameter b_N such that for any fixed $k > 0$

$$(M_i^{(N)} := b_{N^2}^{-1}\tilde{M}_1^{(N)})_{i=1}^k \xrightarrow{(d)} (M_i^{(\infty)})_{i=1}^k, \quad N \to \infty. \tag{2.64}$$

with $M_i^{(\infty)} = (W_1 + \cdots + W_i)^{-\frac{1}{\alpha}}$ and the W_i's are i.i.d. exponential random variables of parameter one. The parameter b_N is explicit:

$$b_N := F^{-1}\left(1 - \frac{1}{N}\right), \tag{2.65}$$

where F is the distribution function in (2.61). Note that F is invertible because ω_x is strictly positive and without atoms. Equivalently we can write

$$b_N := N^{\frac{1}{\alpha}}L(N), \tag{2.66}$$

where L a suitable slowly varying function uniquely defined by L_0 in (2.60). For instance if we consider the Pareto distribution, $F(t) = 1 - t^{-\alpha}$ and therefore $b_N = N^{\frac{1}{\alpha}}$.

Let us rescale the points in Λ_N by considering $(Y_i^{(N)} = \frac{1}{N}\tilde{Y}_i^{(N)})_{i=1}^{N^2}$ as a random set of $[0, 1]^2$. Then in the limit $N \to \infty$, the position of the first k-points is uniformly distributed in $[0, 1]^2$, i.e.,

$$\forall k > 0 \quad (Y_1^{(N)}, \cdots, Y_k^{(N)}) \xrightarrow{(d)} (Y_1^{(\infty)}, \cdots, Y_k^{(\infty)}), \quad N \to \infty, \tag{2.67}$$

where $(Y_i^{(\infty)})_{i\in\mathbb{N}}$ is a sequence of i.i.d. uniform random variables on $[0, 1]^2$.

We have a natural *continuum disorder* in the system, defined by the sequence

$$w = (M_i^{(\infty)}, Y_i^{(\infty)})_{i\in\mathbb{N}}. \tag{2.68}$$

Summarizing, $\tilde{T}^{(N)}$ can be redefined (equivalently at least in distribution) as follows: for any $y, y' \in [0, 1]^2$ we write $y \sim y'$ to mean that y, y' are partially ordered, i.e., $y \le y'$ co-ordinatewise or $y' \le y$ co-ordinatewise. Then

$$\tilde{T}^{(N)} = \max_{A\in C^{(N)}} \sum_{i\in A} \tilde{M}^{(N)}_{Y_i^{(N)}}, \quad N \in \mathbb{N} \cup \{\infty\}, \tag{2.69}$$

where $C^{(N)} = \{A \subset \{1, \cdots, N^d\} : \forall i, j \in A, Y_i^{(N)} \sim Y_j^{(N)}\}$. The final results for the weight function is provided by [50, Theorem 2.1], i.e.,

$$b_{N^2}^{-1}\tilde{T}^{(N)} \xrightarrow{(d)} T^{(\infty)}, \quad N \to \infty.$$

For $N \in \mathbb{N}$ we consider $A^{(N)}$, the set which achieves the maximum in the weight function (2.69). We look at such set as a continuous path: we order all the points of $A^{(N)}$ in an increasing way with respect to the relation $\le$ and we linearly interpolate two consecutive points by an horizontal or vertical segment. We call $P^{(N)}$ the path obtained. Analogously for the limit set $A^{(\infty)}$, we consider $U^{(\infty)}$ defined as the closure of $\cup_i Y_i^{(\infty)} \cup \{(0, \cdots, 0\} \cup \{(1, \cdots, 1\}$. It is conjectured in [50] that such set is connected with probability 1. To work around this, let us observe that there is a.s. a unique way to extend $U^{(\infty)}$ to a connected path, $P^{(\infty)}$, while preserving the increasing path property, i.e. such that for any y, y' in such path it holds $y \sim y'$. To check this fact let us take two consecutive connected components X_1 and X_2 and consider the smallest point of X_1, x_1, which minimizes $x \mapsto \|x\|$ on X_1,and the biggest of X_2, x_2, which maximizes $x \mapsto \|x\|$ on X_2. They are well defined because each X_i is an ordered compact interval. Let us suppose $x_1 \le x_2$, then we can connect x_1 to x_2 only via an horizontal or vertical segment because we cannot have a rectangle between such points, otherwise, with probability 1, we can find a weight on a point in such rectangle respecting the relation $\sim$ with all the points of of $A^{(\infty)}$, in contradiction with its maximality.

Then [50, Theorem 4.4] says that in the space of all closed set of $[0, 1]^2$ equipped with the Hausdorff metric, cf. (2.77), $P^{(N)} \xrightarrow{(d)} P^{(\infty)}$ as $N \to \infty$.

Directed Polymer in random environment with heavy tails

Let us consider a $1 + 1$ - symmetric simple random walk path on the lattice $\frac{\mathbb{N}}{N} \times \frac{\mathbb{N}}{N}$ constrained to come back in 0 after N steps. Let $\mathbf{L}_N$ the set given by the intersection of $\frac{\mathbb{N}}{N} \times \frac{\mathbb{N}}{N}$ with the square $[0, 1]^2$ rotated by $\frac{\pi}{4}$, so that a trajectory of the random walk is a directed path between $(0, 0)$ and $(1, 1)$ on $\mathbf{L}_N$. The reference measure that we perturb is $\hat{\mu}_N$, the uniform one on all possible such trajectories.

Let $(\tilde{M}_i^{(N)})_{i=1,\cdots,|\mathbf{L}_N|}$ the ordered statistic of the environment ω distributed according to (2.60) and $(Y_i^{(N)})_{i=1,\cdots,|\mathbf{L}_N|}$ a random permutation of the points of $\mathbf{L}_N$. The energy of a given trajectory s is

$$\sigma_N(s) = \sum_{i=1}^{|\mathbf{L}_N|} \tilde{M}_i^{(N)} \mathbb{1}\left(Y_i^{(N)} \in s\right). \tag{2.70}$$

In such a way the $\frac{1}{N}$-scaled Gibbs measure is defined like in (2.55):

$$\mu_{\beta,N}(s) = \frac{e^{\beta\sigma_N(s)}}{Z_{\beta,N}} \hat{\mu}_N(s). \tag{2.71}$$

β finite means that we penalize by an entropy cost $E(s)$ a given trajectory. Such entropy is defined as follows: given a 1-Lipschitz function $s : [0, 1] \to [0, 1]$ such that $s(0) = s(1) = 0$ we define

$$E(s) = \int_{-1}^{1} e(s'(x))\mathrm{d}x, \quad e(x) = \frac{1}{2}\left[(1 + x)\log(1 + x) + (1 - x)\log(1 - x)\right]. \tag{2.72}$$

Note that $s'(x)$ is well-defined almost everywhere because $|s'(x)| \le 1$. This function is the rate function in the large deviations principle for the sequence of uniform measures on all possible paths considered, [30, Section 5.1] and it describes precisely the cost associated to visit a set of points, cf. [7, proposition 3.3]: for any fixed set of point ι in $\mathbf{L}_N$ we have, uniformly on N,

$$\hat{\mu}_N(\iota \subset s) = e^{-NE(\hat{\iota})+o(N)}, \quad N \to \infty, \tag{2.73}$$

where $\hat{\iota}$ is the path obtained by linearly interpolating of the consecutive points of ι (which is a 1-Lipschitz function).

We send $\beta \to 0$ as $N \to \infty$ in order to balance the energy, σ_N, and the entropy, E. The interesting regime is

$$\beta_N = \hat{\beta}\frac{N}{b_{N^2}} \overset{(2.66)}{=} \hat{\beta}N^{1-\frac{2}{\alpha}}\frac{1}{L(N^2)}, \quad N \to \infty, \quad \hat{\beta} \ge 0. \tag{2.74}$$

By using such rescaling, if we consider the energy-entropy balance

$$\gamma_{\beta_N}^{(N)} = \arg\max_s \left\{\beta_N \sigma_N^{\omega}(s) - N\, E(s)\right\}, \tag{2.75}$$

the sequence $\gamma_{\beta_N}^{(N)}$ admits a limit in distribution, $\gamma_{\hat{\beta}}^{(\infty)}$, cf. [7, Theorem 2.2]. In analogy with the last passage percolation problem, such limit may be viewed as a random continous path depending on the continuum disorder defined like in (2.68), with the $Y_i^{(\infty)}$'s i.i.d. uniformly on the rotated square. Moreover [7, Theorem 2.2] ensures that the typical trajectories of the simple random walk are concentrated with high probability around $\gamma_{\beta_N}^{(N)}$, that is for any $\delta > 0$

$$\mu_{\beta,N}\left(\left\|s - \gamma_{\beta_N}^{(N)}\right\|_\infty > \delta\right) \overset{\mathbb{P}-\text{probability}}{\underset{N\to\infty}{\longrightarrow}} 0. \tag{2.76}$$

To understand the spatial placement of the typical trajectories it is interesting to study the structure of the limit set $\gamma_{\hat{\beta}}^{(\infty)}$. In [7, proposition 2.5] it has been shown that there exists a critical *random* threshold β_c such that if $\hat{\beta} < \beta_c$, then $\gamma_{\hat{\beta}}^{(\infty)} \equiv 0$, the straight path between $(0,0)$ and $(1,1)$, while if $\beta > \beta_c$, then $\gamma_{\hat{\beta}}^{(\infty)} \not\equiv 0$. One conjectures that for any $\hat{\beta}$ the curve $\gamma_{\hat{\beta}}^{(\infty)}$ is given by a simple path obtained by a linear interpolation of a finite (random) number of points, but this is not proved in the paper [7]. The critical point β_c is partially described by the following theorem

Theorem 2.10. *Let β_c be the critical threshold. Then, denoting by $\mathbb{P}_\infty$ the law of the continuum environment,*

(1) *For any $\alpha \in (0, \frac{1}{3})$, $\beta_c > 0$, $\mathbb{P}_\infty$-a.s.*

(2) *For any $\alpha \in [\frac{1}{2}, 2)$, $\beta_c = 0$, $\mathbb{P}_\infty$-a.s.*

In Section 3.1 we discuss an analogous heavy-tailed pinning model in which the disorder is assumed to satisfy (2.60) with $\alpha \in (0,1)$. The approach used here is close to the one explained for the directed pinning model in a random environment with heavy tails and as byproduct we improve Theorem 2.10, by showing that also for $\alpha \in (1/3, 1/2)$ the critical threshold $\beta_c > 0$, cf. Theorem 3.6.

2.5 Random sets

In this section we are going to present the basic theory of the real random sets, providing an alternative (and fruitful, for the sequel) point of view on the pinning model.

2.5.1 Fell-Matheron topology

A random set is a random variable which takes values in the space of all the closed non-empty subsets of a topological space. We consider such space as a measurable one equipped with the Borel σ-algebra generated by the *Fell-Matheron topology* [36, 67, 69]. We are going to define such topology in the case of the compact metric spaces.

Let (Y, d_Y) be a metric space, then we define the distance between sets, called *Hausdorff distance* d_H, as

$$d_H(A,B) = \sup\left\{\sup_{a\in A}\inf_{b\in B} d_Y(a,b)\,,\ \sup_{b\in B}\inf_{a\in A} d_Y(a,b)\right\}. \tag{2.77}$$

As pictured in figure 2.4, two sets A, B have distance smaller than ε if and only if for any $a \in A$ there exists $b \in B$ such that $d_Y(a,b) < \varepsilon$ and the inverse, by switching the role of A and B. Of course the Hausdorff distance between two sets with the same closure is zero and, in general, the Hausdorff distance is finite if A and B are bounded. In particular on $\mathbf{C}$, the space of all compact non-empty subsets of (Y, d_Y), the Hausdorff distance is a true metric. It is a standard fact, see e.g. [67], that if Y

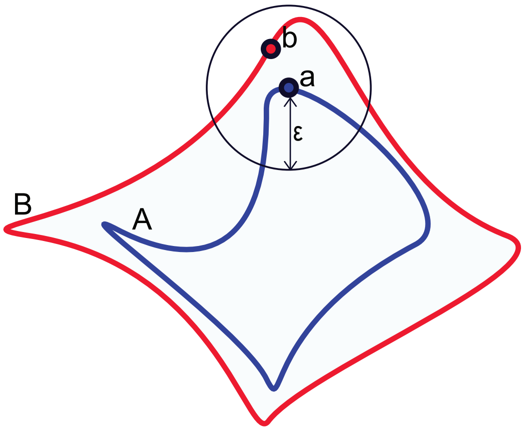

Fig. 2.4: *Hausdorff distance* d_H: two sets A, B have distance smaller than ε if and only if for any $a \in A$ there exists $b \in B$ such that $d(a,b) < \varepsilon$ and the inverse, by switching the role of A and B.

is connected, compact and separable metric space, so is **C**. Moreover the convergence of sequences of sets is completely characterized: a sequence $(A_n)_{n\in\mathbb{N}}$ converges to A if and only if

- for any $x \in A$ there exist elements $x_n \in A_n$ such that $d(x_n, x) \to 0$ as $n \to \infty$,
- every convergent subsequence of points $(x_{n_k})_{k\in\mathbb{N}}$, with $x_{n_k} \in A_{n_k}$, is such that $\lim_{k\to\infty} x_{n_k} \in A$.

Definition 2.11. Let (Y, d_Y) be a compact metric space. Then the topology induced on **C** by the Hausdorff distance is called the *Fell-Matheron topology*.

In the sequel we take $Y = \bar{\mathbb{R}} = [-\infty, \infty]$ and d_Y a metric which makes the space compact. In particular we consider

$$d_{\bar{\mathbb{R}}}(x,y) = |\arctan(x) - \arctan(y)|\,, \quad \arctan(\pm\infty) := \pm\frac{\pi}{2}. \tag{2.78}$$

This choice makes the metric space $(\bar{\mathbb{R}}, d)$ homeomorphic to any compact interval $[a,b]$ equipped with the usual euclidean metric.

For any $A \in \mathbf{C}$ we can define two càdlàg (continue à droite, limitée à gauche) functions $\mathsf{g}(A)$ and $\mathsf{d}(A)$ as follows

$$\mathsf{g}_t(A) = \sup\{x \,:\, x \in A \cup [-\infty, t]\} \quad \mathsf{d}_t(A) = \inf\{x \,:\, x \in A \cup (t, \infty]\}\,. \tag{2.79}$$

These functions characterize completely the set A, i.e.

$$A = \{t \in \bar{\mathbb{R}} \,:\, \mathsf{g}_t(A) = t\} = \{t \in \bar{\mathbb{R}} \,:\, \mathsf{d}_{t^-}(A) = t\} \tag{2.80}$$

and convergence of closed sets $(A_n)_{n\in\mathbb{N}}$ in the Hausdorff metric is equivalent to the convergence of the corresponding functions $(\mathsf{g}(A_n))_{n\in\mathbb{N}}$ in the Skorokhod metric [24, Remark A.5]. The Skorokhod metric ρ is the natural metric on the space of the càdlàg functions that generalizes the uniform one, in sense that if f, g are continuum function, then $\rho(f,g) = \|f - g\|_\infty$. For more details see [13, 53].

2.5.2 Characterization of random closed sets

Expression (2.80) suggests that the law of a random set $(\mathcal{A}, \mathrm{P})$ can be described in terms of the two random functions $\mathsf{g}_t(\mathcal{A})$ and $\mathsf{d}_t(\mathcal{A})$: we call the family of laws of $(\mathsf{g}_{t_i}(\mathcal{A}), \mathsf{d}_{t_i}(\mathcal{A}))_{i=1,\cdots,k}$ of varying $0 < t_1 < \cdots < t_k$ the *finite dimensional distributions (f.d.d.) of* $\mathcal{A}$,which define uniquely the law of $\mathcal{A}$.

Similarly, a sequence of random sets $(\mathcal{A}_n)_{n\in\mathbb{N}}$ converges in law if and only if the sequence of f.d.d. $((\mathsf{g}_{t_i}(\mathcal{A}_n), \mathsf{d}_{t_i}(\mathcal{A}_n))_{i=1,\cdots,k})_{n\in\mathbb{N}}$ has a limit in law for any fixed $t_1 < \cdots < t_k$ in some dense set $\mathcal{T}$. In such case the law of the limit of $(\mathcal{A}_n)_{n\in\mathbb{N}}$ is defined by the limit of the f.d.d. See [24, Appendix A] for more details.

Remark 2.12. To characterize the law of a random set $\mathcal{A}$ it is enough to consider a *restricted* version of the f.d.d., which is very useful in the computations. To introduce such object we consider the family of events

$$E^{\mathcal{A}}_{t_1,\cdots,t_k} := \{\mathcal{A} \cap (t_i, t_{i+1}] \neq \emptyset, \ i = 1, \cdots, k-1\} \tag{2.81}$$

and we define the *restricted f.d.d.* of $\mathcal{A}$ as the f.d.d. restricted on the event $E^{\mathcal{A}}_{t_1,\cdots,t_k}$, with indices $t_1 < \cdots < t_k$ in some dense subset $\mathcal{T} \subset R_+$. Note that the condition $\mathcal{A} \cap (s,t] \neq \emptyset$ means $s \le \mathsf{d}_s(\mathcal{A}) \le \mathsf{g}_t(\mathcal{A}) \le t$.

2.5.3 Renewal processes

In this section we focus our attention on the renewal process, a particular random subset of $\mathbb{N}_0$ introduced in Section 2.2. We recall that a process $(\tau = (\tau_n)_{n\in\mathbb{N}_0}, \mathrm{P})$ is called *renewal process* if $\tau_0 = 0$ and the sequence of *inter-arrival times* $(\tau_n - \tau_{n-1})_{n\in\mathbb{N}}$ is i.i.d. We to look at τ as a random variable in the space $(\mathbf{C}, d_H)$ and the i.i.d. structure of the inter-arrival times implies that if for $n \in \mathbb{N}$ we consider $\mathcal{F}_n$ the filtration generated by $\tau \cap [0,n]$, then for every $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$-stopping time λ such that $\lambda \in \tau$, P-a.s., the translate random set $(\tau - \lambda) \cap [0,\infty)$ under P is independent of $\mathcal{F}_\lambda$ and it is distributed as τ.

According to Section 2.5.2, the structure of τ is determined by the functions $\mathsf{g}_t(\tau)$ and $\mathsf{d}_t(\tau)$, with $t \in \mathbb{N}$. The joint law of $\mathsf{g}_t(\tau)$ and $\mathsf{d}_t(\tau)$ is explicitly described by the *law of* τ_1, $K(n) := \mathrm{P}(\tau_1 = n)$, and by the *renewal function* $u(n) := \mathrm{P}(n \in \tau)$. Indeed for any $r, s \in \mathbb{N} \,:\, r \le t < s$ we have

$$\mathrm{P}\left(\mathsf{g}_t(\tau) = r, \ \mathsf{d}_t(\tau) = s\right) = u(r)K(s-r).$$

Under the assumption (2.5) on $K(\cdot)$ the *renewal Theorem* assets (cf. [6, Chapter 1, Theorem 2.2] or [14, Theorem 8.7])

$$\lim_{N\to\infty} \mathrm{P}(N \in \tau) = \frac{1}{\mathrm{E}(\tau_1)}, \quad \text{where } 1/\infty := 0. \tag{2.82}$$

The precise and minimal assumption on $K(\cdot)$ is the *aperiodicity*, that is the greatest common divisor (g.c.d.) of $\{p \,:\, K(n) = 0, \ \forall n \in p\mathbb{N}\}$ is 1. Anyway any p-periodicity is straightened out by restricting the limit on the lattice $p\mathbb{N}$ and (2.82) holds by replacing the r.h.s. with $p/E(\tau_1)$, see [6, Chapter 1, Corollary 2.3]. When $\mathrm{E}(\tau_1) = \infty$ it is interesting to study the asymptotics of $\mathrm{P}(N \in \tau)$ as $N \to \infty$. Note that $\mathrm{E}(\tau_1) = \infty$ happens when we consider terminating ($K(\infty) > 0$) renewal processes, or non-terminating one with non-integrable tails, like the renewal process defined by the symmetric simple random walk, in which $K(n) \sim \mathtt{const}\, n^{-3/2}$.

The key word to discuss the terminating case is *sub-exponentiality*, which means

$$\begin{aligned} &\lim_{n\to\infty} K(n+k)/K(n) = 1, \quad \forall\, k > 0, \\ &\lim_{n\to\infty} K^{*(2)}(n)/K(n) = 2, \end{aligned} \tag{2.83}$$

(see Appendix 4.A). Note that $K^{*(2)}(n) = \mathrm{P}(\tau_2 = n)$ is the law of the second point of the renewal process. These conditions are satisfied if (cf. Section 4.A.1 below and [44, Lemma A.5])

$$K(n) \sim \frac{L(n)}{n^\rho} e^{-\mathtt{c} n^\gamma}, \quad n \to \infty, \tag{2.84}$$

with $L(\cdot)$ a slowly varying function, $\gamma \in [0,1)$ and $\rho \in \mathbb{R}$ if $\gamma > 0$, otherwise ρ must be strictly positive. Whenever $\gamma > 0$, we say that $K(\cdot)$ is stretched-exponential, while $\gamma = 0$ and $\rho > 1$ corresponds to the heavy-tailed case consider in Section 2.2 and 2.3.

Lemma 2.13. *Let (τ, P) be a renewal process, then,*

- *If (τ, P) is terminating and sub-exponential, cf. Theorem 4.41 and [44, Theorem A.4],*

$$\mathrm{P}(N \in \tau) \underset{N\to\infty}{\sim} \frac{K(N)}{K(\infty)^2}. \tag{2.85}$$

- *If (τ, P) is non-terminating and with (non-integrable) heavy tails, i.e.,*

$$K(N) \underset{N\to\infty}{\sim} \frac{L(N)}{N^{1+\alpha}}, \quad \alpha \in (0,1), \tag{2.86}$$

its asymptotics is provided by Doney [34]:

$$\mathrm{P}(N \in \tau) \underset{N\to\infty}{\sim} \frac{\alpha \sin(\pi\alpha)}{\pi} \frac{1}{L(N)N^{1-\alpha}}. \tag{2.87}$$

2.5.4 Scaling limits of renewal processes: the Regenerative set

For any $N \in \mathbb{N}$ we consider the *rescaled* non-terminating ($K(\infty) = 0$) renewal process $\tau/N = \{0, \tau_1/N, \tau_2/N, \cdots\}$ and the random sequence of subsets $(\tau/N)_{N\in\mathbb{N}}$. Under the assumptions (2.3) for some $\alpha \in (0,1)$, such sequence $(\tau/N)_{N\in\mathbb{N}}$ admits an explicit limit in distribution, called *regenerative set* $\boldsymbol{\tau}^\alpha$ [11, 12]. It is an *universal limit* depending only on the coefficient $\alpha \in (0,1)$ and it is independent of the fine details of the renewal process, like the choice of the slowly varying function. The law of the regenerative set is defined through its restricted f.d.d.: for any $0 = y_0 \le x_1 \le t_1 < y_1 \le \cdots \le x_k \le t_k < y_k$

$$\begin{aligned} &\frac{\mathrm{P}\left(\mathsf{g}_{t_1}(\boldsymbol{\tau}^\alpha) \in \mathrm{d}x_1, \mathsf{d}_{t_1}(\boldsymbol{\tau}^\alpha) \in \mathrm{d}y_1, \cdots \mathsf{g}_{t_k}(\boldsymbol{\tau}^\alpha) \in \mathrm{d}x_k, \mathsf{d}_{t_k}(\boldsymbol{\tau}^\alpha) \in \mathrm{d}y_k\right)}{\mathrm{d}x_1 \mathrm{d}y_1 \cdots \mathrm{d}x_k \mathrm{d}y_k} \\ &= \prod_{i=1}^k \frac{C_\alpha}{(x_i - y_{i-1})^{1-\alpha}(y_i - x_i)^{1+\alpha}} =: \mathrm{f}^{(\alpha)}_{t_1,\cdots,t_k}(x_1, y_1, \cdots, x_k, y_k), \\ &\text{where } C_\alpha = \frac{\alpha \sin(\alpha\pi)}{\pi}. \end{aligned} \tag{2.88}$$

Remark 2.14. As particular case we can deduce the joint distribution of $(\mathsf{g}_t(\boldsymbol{\tau}^\alpha), \mathsf{d}_t(\boldsymbol{\tau}^\alpha))$. Let P_x be the law of a delayed regenerative set started form x, that is $\mathrm{P}_x(\boldsymbol{\tau}^\alpha \in \cdot) := \mathrm{P}(\boldsymbol{\tau}^\alpha + x \in \cdot)$, then the joint distribution of $(\mathsf{g}_t(\boldsymbol{\tau}^\alpha), \mathsf{d}_t(\boldsymbol{\tau}^\alpha))$ is given by

$$\frac{\mathrm{P}_x\left(\mathsf{g}_t(\boldsymbol{\tau}^\alpha) \in du, \mathsf{d}_t(\boldsymbol{\tau}^\alpha) \in dv\right)}{dudv} = C_\alpha \frac{\mathbb{1}_{u\in(x,t)}\mathbb{1}_{v\in(t,\infty)}}{(u-x)^{1-\alpha}(v-u)^{1+\alpha}}. \tag{2.89}$$

from which we can deduce the distribution of $\mathsf{g}_t(\boldsymbol{\tau}^\alpha)$:

$$\frac{\mathrm{P}_x\left(\mathsf{g}_t(\boldsymbol{\tau}^\alpha) \in \mathrm{d}u\right)}{\mathrm{d}u} = \frac{C_\alpha}{\alpha} \frac{\mathbb{1}_{u\in(x,t)}}{(u-x)^{1-\alpha}(t-u)^\alpha}. \tag{2.90}$$

Analogously, for any $T > 0$ we can consider the *conditioned renewal set* always defined by its restricted f.d.d.: for any $0 = y_0 \le x_1 \le t_1 < y_1 \le \cdots \le x_k \le t_k < y_k < T$

$$\begin{aligned} &\frac{\mathrm{P}^T\left(\mathsf{g}_{t_1}(\boldsymbol{\tau}^\alpha) \in \mathrm{d}x_1, \mathsf{d}_{t_1}(\boldsymbol{\tau}^\alpha) \in \mathrm{d}y_1, \cdots \mathsf{g}_{t_k}(\boldsymbol{\tau}^\alpha) \in \mathrm{d}x_k, \mathsf{d}_{t_k}(\boldsymbol{\tau}^\alpha) \in \mathrm{d}y_k\right)}{\mathrm{d}x_1 \mathrm{d}y_1 \cdots \mathrm{d}x_k \mathrm{d}y_k} \\ &= \frac{T^{1-\alpha}}{(T-y_k)^{1-\alpha}} \prod_{i=1}^k \frac{C_\alpha}{(x_i - y_{i-1})^{1-\alpha}(y_i - x_i)^{1+\alpha}} =: \mathrm{f}^{(\alpha),\mathrm{c}}_{t_1,\cdots,t_k,T}(x_1, y_1, \cdots, x_k, y_k). \end{aligned} \tag{2.91}$$

The conditioned regenerative set is the limit of the rescaled *conditioned* renewal process: for any $T > 0$ the rescaled renewal process $\tau/N \cap [0,T]\}_{N\in\mathbb{N}}$ conditioned to visit $\frac{\lfloor NT \rfloor}{N}$ converges to the *conditioned renewal set*. The proof consists in writing explicitly the f.d.d. of the rescaled renewal process $(\mathsf{g}_{t_i}(\tau/N), \mathsf{d}_{t_i}(\tau/N))_{i=1,\cdots,k}$ and a direct computation based on a Riemann sum argument shows the convergence. Let us give a sketch of the proof: for any $0 \le x_1 < t_1 < y_1 < \cdots < x_k < t_k < y_k \le 1$ we have

$$\begin{aligned}
&\lim_{N\to\infty} N^{2k}\, \mathrm{P}\left(\mathsf{g}_{t_1}(\tau/N) = x_1, \mathsf{d}_{t_1}(\tau/N) = y_1, \cdots, (\mathsf{g}_{t_k}(\tau/N) = x_k, \mathsf{d}_{t_k}(\tau/N) = y_k, \middle| N \in \tau\right)\\
&= \lim_{N\to\infty} N^{2k} \left[\prod_{i=1}^{k} u(N(x_i - y_{i-1})K(N(y_i - x_i))\right] \frac{u(N(1-y_k))}{u(N)}\\
&= \frac{1}{(1-y_k)^{1-\alpha}} \prod_{i=1}^{k} \frac{C_\alpha}{(x_i - y_{i-1})^{1+\alpha}(y_i - x_i)^{1-\alpha}}.
\end{aligned}$$

where $y_0 := 0$. In particular we observe that the value of the limit is the density of the finite dimensional distributions of the conditioned regenerative set. A dominated convergence argument provides the conclusion of the proof. The detailed proof can be found in [24, Appendix A].

Finally like the renewal process, we stress that also the regenerative set enjoys the renewal property: for $u \ge 0$ we denote by $\mathcal{G}_u$ the filtration generated by $\boldsymbol{\tau}^\alpha \cap [0, u]$. Then $\boldsymbol{\tau}^\alpha$ enjoys the *regenerative property*: for every $\{\mathcal{G}_u\}_{u\ge 0}$-stopping time σ such that $\sigma \in \boldsymbol{\tau}^\alpha$, P_x-a.s., the translate random set $(\boldsymbol{\tau}^\alpha - \sigma) \cap [0, \infty)$ under P is independent of $\mathcal{G}_\sigma$ and it is distributed as $\boldsymbol{\tau}^\alpha$. An important example of such stopping time is the random function $\mathsf{d}_t(\boldsymbol{\tau}^\alpha)$, for some fixed $t > 0$. Let us stress that the deduction of such property is obtained by working directly on the regenerative set and its characterization as closure of the image of an α-stable subordinator. This point of view is not treated in this thesis, for more details about we refer to [11, 12, 22].

2.5.5 Convergence of the pinning model

We want to conclude this section by considering the pinning model introduced in section 2.2 and 2.3 in the light of the theory of the random sets, which will be very useful for the sequel. We are going to discuss again the localization/de-localization phenomena that the pinning model shows with the language of the random sets.

For this purpose we consider the sequence of random sets $\tau_{(N)} = \{\tau/N \cap [0,1]\}_{N\in\mathbb{N}}$. We consider the space of all closed subsets of $[0,1]$ as a topological space equipped with the Fell-Matheron topology introduced in Section 2.5.1. The results that we are going to present are [44, Theorem 2.5 and theorem 2.7]:

Theorem 2.15. *Let us consider the pinning model $(\tau, \mathrm{P}^{\mathtt{a}}_{N,h})$, with*

$$\mathrm{P}^{\mathtt{a}}_{N,h}(\tau) = \frac{\exp\left\{h \sum_{n=1}^{N} \mathbb{1}_{n\in\tau}\right\}}{\mathrm{Z}^{\mathtt{a}}_h(N))} \Phi^{\mathtt{a}}_N\, \mathrm{P}(\tau), \tag{2.92}$$

where $\Phi^{\mathtt{f}}_N(\cdot) = 1$ defines the free model and $\Phi^{\hat{\mathtt{c}}}_N(\cdot) = \mathbb{1}_{N\in\tau}$ the constrained one. Let us assume that $K(\cdot)$ has heavy-tails, i.e.,

$$K(N) \sim \frac{L(N)}{N^{1+\alpha}}, \quad N \to \infty, \quad \alpha > 0, \tag{2.93}$$

and $K(n) > 0$ for any $n \in \mathbb{N}$. Then $h_c = -\log(1 - K(\infty))$ and

(1) *if $h > h_c$, then $\tau_{(N)} \xrightarrow{(\mathrm{d})} [0,1]$, as $N \to \infty$*

(2) *if $h < h_c$, then $\tau_{(N)} \xrightarrow{(\mathrm{d})} \begin{cases} \{0,1\}, & \mathtt{a} = \hat{\mathtt{c}}, \\ \{0\}, & \mathtt{a} = \mathtt{f}, \end{cases}$ as $N \to \infty$*

(3) *In the free case ($\mathtt{a} = \mathtt{f}$) if $h = h_c$ and $K(\infty) = 0$, then $\tau_{(N)} \xrightarrow{(\mathrm{d})} \tau^{(\alpha\wedge 1)} \cap [0,1]$, a regenerative set of*

exponent $\alpha \wedge 1$, where $\tau^{(1)} = [0,\infty)$ is the degenerate case.
If $K(\infty) > 0$, then for $\alpha \in (0,1)$ the sequence $\tau_{(N)}$ converges in law to a random set $\hat{\tau}^\alpha$ absolutely continuous with respect to $\tau^{(\alpha)} \cap [0,1]$ and the Radon-Nikodym derivative is explicit: $(\alpha\pi/\sin(\alpha\pi))(1-\max\{\boldsymbol{\tau}^\alpha \cap [0,1]\})$, otherwise if $\alpha > 1$, then $\tau_{(N)}$ converges in law to $[0,U]$, where U is an uniform random variable on $[0,1]$.

Note that in (3) the condition $K(\infty) = 0$ implies $h_c = 0$ and the statement is nothing but the content of section 2.5.4. We stress that (3) can be extended to the constrained case. To be more precise we take $K(\infty) = 0$ ($h = h_c = 0$), $\alpha \in (0,1)$ and we note that $\mathrm{P}^{\hat{\mathtt{c}}}_{N,h_c=0}$ is the pinning model in (2.11), which is nothing but the law of the original renewal process conditioned to visit N. According to Section 2.5.4, the sequence $\tau_{(N)}$ conditioned to visit 1 converges to (2.91), the regenerative set $\boldsymbol{\tau}^\alpha \cap [0,1]$ conditioned to visit 1. The same results holds if 1 is replaced by some $T > 0$.

We have an analogous statement of Theorem 2.15 in the stretched-exponential case (2.83):

Proposition 2.16. *If (τ, P) is non-terminating, then the behavior of the homogeneous pinning model $(\tau, \mathrm{P}^{\mathtt{a}}_{N,h})$, with $\mathtt{a} = \mathtt{f}$ or $\hat{\mathtt{c}}$ is the following*

- *if $h \geq 0$, then $\tau_{(N)} \xrightarrow{(\mathrm{d})} [0,1]$,*
- *if $h < 0$, then $\tau_{(N)} \xrightarrow{(\mathrm{d})} \begin{cases} \{0,1\}, & \mathtt{a} = \hat{\mathtt{c}}, \\ \{0\}, & \mathtt{a} = \mathtt{f}. \end{cases}$*

The proof runs similarly to the one of Theorem 2.15.

Theorem 2.15 describes the behavior of the homogeneous pinning model when h is fixed. We can consider also the model in which the parameter h goes to 0 as N grows to ∞ in order to have in the limit a weak-perturbation of the original law. The interesting result for the homogeneous pinning model was provided by Sohier [76] and then generalized by [24, Theorem 1.3], leading to define the *continuum disorder pinning model* and the homogeneous one is the particular case in which we switch off the continuum disorder. In the sequel we are going to explain (not formally) how the continuum pinning model appears naturally as limit of the pinning model. Let us stress that the analysis of the disordered version requires essentially the same techniques of the homogeneous one, because the definition of the continuum pinning model is based only on the existence of a continuum limit for the partition function. In the sequel we discuss directly the disordered pinning model. The idea is to study the convergence of the restricted f.d.d. of the conditional pinning model, cf. Section 2.5.2: let us fix $0 \leq a < t < b \leq 1$ and consider $\beta = \beta_N$, $h = h_N$ as in (2.48) then, by using the convergence of the partition function showed in [24, Theorem 2.1],

$$
\begin{aligned}
&\mathrm{P}^{\mathtt{c}}_{N,\beta_N,h_N}\left(\mathtt{g}_t(\tau/N) \in [a, a+\varepsilon]\,,\ \mathtt{d}_t(\tau/N) \in [b, b+\varepsilon] \,|\, N \in \tau\right) \\
&= \sum_{\substack{x \in [a,a+\varepsilon] \cap \frac{\mathbb{N}}{N} \\ y \in [b,b+\varepsilon] \cap \frac{\mathbb{N}}{N}}} \frac{Z^{\omega,\mathtt{c}}_{\beta_N,h_N}(0,Nx) e^{\beta_N \omega_y + h} Z^{\omega,\mathtt{c}}_{\beta_N,h_N}(N(y-x),N)}{Z^{\omega,\mathtt{c}}_{\beta_N,h_N}(N)} \frac{u(Nx)K(N(y-x))u(N(1-y))}{u(N)} \\
&\approx \frac{1}{N^2} \sum_{\substack{x \in [a,a+\varepsilon] \cap \frac{\mathbb{N}}{N} \\ y \in [b,b+\varepsilon] \cap \frac{\mathbb{N}}{N}}} \frac{Z^{\omega,\mathtt{c}}_{\beta_N,h_N}(0,Nx) Z^{\omega,\mathtt{c}}_{\beta_N,h_N}(N(y-x),N)}{Z^{\omega,\mathtt{c}}_{\beta_N,h_N}(N)} \frac{1}{x^{1-\alpha}(y-x)^{1+\alpha}(1-y)^{1-\alpha}} \\
&\overset{N\to\infty}{\to} \int_a^{a+\varepsilon}\int_b^{b+\varepsilon} \frac{\mathbf{Z}^{W,\mathtt{c}}_{\hat{\beta},\hat{h}}(0,x) \mathbf{Z}^{W,\mathtt{c}}_{\hat{\beta},\hat{h}}((y-x),1)}{\mathbf{Z}^{W,\mathtt{c}}_{\hat{\beta},\hat{h}}(1)} \frac{1}{x^{1-\alpha}(y-x)^{1+\alpha}(1-y)^{1-\alpha}} \mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

We use this last expression to define the *continuum pinning model*, namely

$$\mathsf{P}^{\mathsf{c}}_{1,\hat\beta,\hat h}\left(\mathsf{g}_t(\boldsymbol{\tau}^\alpha)\in[a,a+\varepsilon]\,,\ \mathsf{d}_t(\boldsymbol{\tau}^\alpha)\in[b,b+\varepsilon]\right):=$$
$$\int_a^{a+\varepsilon}\int_b^{b+\varepsilon}\frac{\mathbf{Z}^{W,\mathsf{c}}_{\hat\beta,\hat h}(0,x)\mathbf{Z}^{W,\mathsf{c}}_{\hat\beta,\hat h}(y-x,1)}{\mathbf{Z}^{W,\mathsf{c}}_{\hat\beta,\hat h}(1)}\frac{1}{x^{1-\alpha}(y-x)^{1+\alpha}}\mathrm{d}x\mathrm{d}y.$$

The general result is provided by [24, Theorem 1.3 & Theorem 1.6]: for $T>0$ let $\mathrm{f}^{(\alpha),T}_{t_1,\cdots,t_k,T}(x_1,y_1,\cdots,x_k,y_k)$ be the density of the conditional regenerative set on T, (2.91), then the *continuum disordered pinning model* is the unique probability measure defined by the following restricted f.d.d.: for any $0=y_0\le x_1\le t_1<y_1\le\cdots\le x_k\le t_k<y_k<T$

$$\frac{\mathsf{P}^{\mathsf{c}}_{T,\hat\beta,\hat h}\left(\mathsf{g}_{t_1}(\boldsymbol{\tau}^\alpha)\in\mathrm{d}x_1,\mathsf{d}_{t_1}(\boldsymbol{\tau}^\alpha)\in\mathrm{d}y_1,\cdots\mathsf{g}_{t_k}(\boldsymbol{\tau}^\alpha)\in\mathrm{d}x_k,\mathsf{d}_{t_k}(\boldsymbol{\tau}^\alpha)\in\mathrm{d}y_k\right)}{\mathrm{d}x_1\mathrm{d}y_1\cdots\mathrm{d}x_k\mathrm{d}y_k}:=$$
$$\mathrm{f}^{(\alpha),T}_{t_1,\cdots,t_k,T}(x_1,y_1,\cdots,x_k,y_k)\frac{\prod_{i=1}^{k+1}\mathbf{Z}^{W,\mathsf{c}}_{\hat\beta,\hat h}(x_{i-1},y_i)}{\mathbf{Z}^{W,\mathsf{c}}_{\hat\beta,\hat h}(0,T)},\tag{2.94}$$

where $x_0:=0$ and $y_{k+1}:=T$. Such continuum pinning model is the limit of the discrete one, i.e., for any fixed $T>0$ we can consider $P^\omega_{\lfloor NT\rfloor,\beta_N,h_N}(\tau/N\in\cdot\,|\,\lfloor NT\rfloor\in\tau)$ as a sequence of probability measure in the space of Borel probability measures on $\mathbf{C}$, $\mathcal{M}(\mathbf{C})$, then it converges in distribution to the continuum disordered pinning model $\mathsf{P}^{\mathsf{c}}_{T,\hat\beta,\hat h}(\boldsymbol{\tau}^\alpha\in\cdot)$.

3 Results of the Thesis

In this chapter we present the results obtained throughout the thesis. In the following two sections we give the precise assumptions about the models considered and then our results. Let us stress that such results are described in the pre-prints [78] and [25], that are based on the contents of Chapters 4 and 5, to which we refer for the proofs and the technical details.

3.1 The Pinning Model with heavy tailed disorder

In this section we discuss the results of this thesis obtained for a disordered pinning model with *heavy-tailed disorder*, (2.60), with exponent $\alpha \in (0,1)$. In this section we consider a renewal process τ with *stretched-exponential* inter-arrival distribution, (2.83), with exponent $\gamma \in (0,1)$. We prove that by sending suitably to 0 the parameter β — which tunes the force of the disorder — as N grows to ∞, then the rescaled renewal process $\tau/N \cap [0,1]$ shows concentration around a deterministic set which depends on a quenched realization of the disorder. Moreover for any α there exists a critical threshold of β below which the effect of the disorder changes macroscopically the behavior of the model.

3.1.1 Definition of the model

According to Section 2.5.5 for any fixed $N \in \mathbb{N}$ we consider $\tau/N \cap [0,1] = \{\tau_j/N : \tau_j \leq N\}$, the rescaled renewal process up to time N and we denote by P_N the law of $\tau/N \cap [0,1]$, which turns out to be a probability measure on the space of all subsets of $\{0, 1/N, \cdots, 1\}$. The *pinning model* $\mathrm{P}^{\omega}_{\beta,h,N}$ is the probability measure defined by the following Radon-Nikodym derivative

$$\frac{\mathrm{dP}^{\omega}_{\beta,h,N}}{\mathrm{dP}_N}(I) = \frac{1}{\mathrm{Z}^{\omega}_{\beta,h,N}} \exp\left(\sum_{n=1}^{N-1} (\beta\omega_n + h)\mathbb{1}(n/N \in I)\right)\mathbb{1}(1 \in I), \tag{3.1}$$

where, we recall, $\beta \geq 0$, $h \in \mathbb{R}$ and the sequence $\omega = (\omega_i)_{i\in\mathbb{N}}$ is the disorder.

We look at the pinning model as a *random* probability measure on the space $\mathbf{X}$ of all closed subsets of $[0,1]$ which contain both 0 and 1,

$$\mathbf{X} = \{I \subset [0,1] : I \text{ is closed and } 0, 1 \in I\}, \tag{3.2}$$

with support given by $\mathbf{X}^{(N)}$, the set of all subsets of $\{0, 1/N, \cdots, 1\}$ containing both 0 and 1. Note that definition (3.1) is equivalent to the original one (2.24).

3.1.2 Disorder and energy

Through this section the disorder of the model of size N is a finite i.i.d. sequence $\omega = (\omega_1, \cdots, \omega_{N-1})$ of random variables whose tail is regularly varying with index $\alpha \in (0,1)$, namely

$$\mathbb{P}(\omega_1 > t) \sim L_0(t)\, t^{-\alpha}, \quad t \to \infty, \tag{3.3}$$

where $\alpha \in (0,1)$ and $L_0(\cdot)$ is a slowly varying function, cf. [14]. Moreover we assume that the law of ω_1 has no atom and it is supported in $(0,\infty)$, i.e. ω_1 is a positive random variable. We write the disorder in terms of its ordered statistics $(\tilde{M}^{(N)}_i, Y^{(N)}_i)_{i=1}^{N-1}$ presented in Section 2.4.3: we recall that $(\tilde{M}^{(N)}_i)_{i=1}^{N-1}$ is the ordered statistics of the ω_i's, i.e., $\tilde{M}^{(N)}_1$ is the maximum value among $(\omega_1, \cdots, \omega_{N-1})$, $\tilde{M}^{(N)}_2$ the second one and so on, while $(Y^{(N)}_i)_{i=1}^{N-1}$ is a random permutation of the points $\{\frac{1}{N}, \cdots 1 - \frac{1}{N}\}$ and each $Y^{(N)}_i$ defines the position of $\tilde{M}^{(N)}_i$ among the points $\{\frac{1}{N}, \cdots, 1 - \frac{1}{N}\}$. We define the energy σ_N

of a given realization of the renewal process I as

$$\sigma_N(I) = \sum_{i=1}^{N-1} \tilde{M}_i^{(N)} \mathbb{1}(Y_i^{(N)} \in I), \tag{3.4}$$

3.1.3 Renewal process and entropy

The renewal process considered, τ, is taken to be *non-terminating* and satisfying the following assumptions

(1) subexponential: $\lim_{n\to\infty} K(n+k)/K(n) = 1$ for any $k > 0$ and $\lim_{n\to\infty} K^{*(2)}(n)/K(n) = 2$ (see Appendix 4.A),

(2) stretched-exponentiality: $\lim_{n\to\infty} \log K(n)/N^{\gamma} = -\mathsf{c}$, for a suitable constant $\mathsf{c} > 0$ and $\gamma \in (0, 1)$,

(3) $K(n) > 0$ for any $n \in \mathbb{N}$.

For instance (cf. Section 4.A) these conditions are satisfied if

$$K(n) \sim \frac{L(n)}{n^{\rho}} e^{-\mathsf{c} n^{\gamma}}, \quad n \to \infty, \tag{3.5}$$

with $\rho \in \mathbb{R}$, $\gamma \in (0, 1)$ and $L(\cdot)$ a slowly varying function. We recall that in such case the behavior of the renewal process is given by proposition 2.16.

3.1.4 Main results

Our aim is to study the behavior of $\tau/N \cap [0, 1]$ under the pinning model $\mathrm{P}^{\omega}_{\beta,h,N}$ when N is large. Let us stress that we are using a positive disorder, so that $h \geq 0$ in the definition (3.1) forces $\tau/N \cap [0, 1]$ to converge to $[0, 1]$. This is a consequence of Proposition 2.16. To have a non-trivial behavior of the model we may consider a terminating renewal process, therefore, by proposition 2.16, we choose and fix $h < 0$ in (3.1). Such choice is equivalent to consider $h = 0$ in the exponent of (3.1) and replace the law of the renewal process by a terminating one, with $K(\infty) = 1 - e^{h} > 0$. Let us explain why this allows to define an entropy of the system which competes with the energy (3.4): the probability to visit a given set of ℓ points (with ℓ independent of N) $\iota = \{0 = \iota_0 < \iota_1 < \cdots < \iota_\ell = 1\} \in \mathbf{X}^{(N)}$ is provided by Lemma 2.13, (2.85):

$$\mathrm{P}_N(\iota \subset I) := \prod_{i=1}^{\ell} \mathrm{P}\left(N(\iota_i - \iota_{i-1}) \in \tau\right) \overset{(2.85)}{\approx} e^{-\mathsf{c}N^{\gamma} \sum_{i=1}^{\ell}(\iota_i - \iota_{i-1})^{\gamma} + o(N^{\gamma})}. \tag{3.6}$$

In analogy with (2.73) we define the entropy $E(\iota)$ as

$$E(\iota) = \sum_{i=1}^{\ell} (\iota_i - \iota_{i-1})^{\gamma}. \tag{3.7}$$

Remark 3.1. It turns out that $E : \mathbf{X}^{(N)} \to \mathbb{R}_+$ is a lower semi continuous and thus it admits a *minimal* lower semicontinous extension to the whole $\mathbf{X}$, see Section 4.2.3. Therefore we can define $E(\iota)$ for any possible set $\iota \in \mathbf{X}$.

In analogy with the directed polymer in a random environment with heavy tails, cf. Section 2.4.3, we rescale β by sending it to 0 as N grows to ∞ in order to balance the energy and the entropy. The interesting regime is the following:

$$\beta_N \sim \hat{\beta} N^{\gamma - \frac{1}{\alpha}} \frac{1}{L(N)}, \quad N \to \infty, \tag{3.8}$$

with L as in (2.66). By using such rescaling, if we consider the set $I_{\beta_N,N}$ which maximizes the difference between the energy and the entropy on all the possible realizations of the rescaled renewal process

$$I_{\beta_N,N} = \arg\max_{I\in\mathbf{X}^{(N)}} \{\beta_N \sigma_N(I) - \mathrm{c}N^{\gamma} E(I)\}, \tag{3.9}$$

then with high $\mathbb{P}$-probability $\left(\tau/N \cap [0,1], \mathrm{P}^{\omega}_{h,\beta_N,N}\right)$ is concentrated in the Hausdorff distance (2.77) around the set $I_{\beta_N,N}$, which depends only on the disorder, see figure 3.1. The precise result is the following

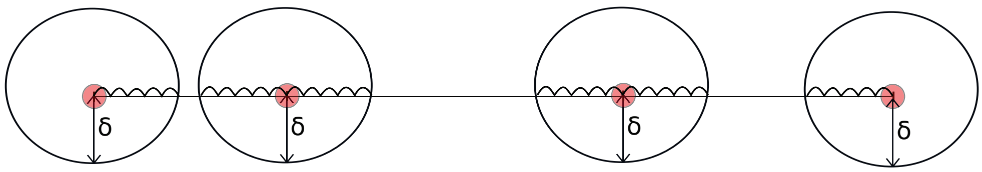

Fig. 3.1: In red we have marked the points of $I_{\beta_N,N}$. Then given $\delta > 0$, if N is large enough, then with high $\mathbb{P}$-probability, with respect to the pinning model, all the points of $\tau/N \cap [0,1]$ are contained in a δ - neighborhood of $I_{\beta_N,N}$ in the Hausdorff metric.

Theorem 3.2. *Let $(\beta_N)_N$ be as in (3.8). For any $N \in \mathbb{N}$, $\hat{\beta} > 0$ consider the random set $I_{\beta_N,N}$ defined in* (3.9)*. Then, for any $\delta > 0$, $h < 0$ one has that $\mathrm{P}^{\omega}_{\beta_N,h,N}\left(d_H(I, I_{\beta_N,N}) > \delta\right)$ converges to 0 as $N \to \infty$ in probability (with respect to the disorder ω). More precisely for any $\varepsilon > 0$ there exists $\nu = \nu(\varepsilon,\delta)$ and $\hat{N}$ such that for all $N > \hat{N}$*

$$\mathbb{P}\left(\mathrm{P}^{\omega}_{\beta_N,h,N}\left(d_H(I, I_{\beta_N,N}) > \delta\right) < e^{-\nu N^{\gamma}}\right) > 1 - \varepsilon. \tag{3.10}$$

In the second result we control the convergence in law of $I_{\beta_N,N}$. For this purpose, according to the result about the convergence of the ordered statistics (2.64) and (2.67), we define the *continuum disorder* as in (2.68), i.e.,

$$w = (M_i^{(\infty)}, Y_i^{(\infty)})_{i\in\mathbb{N}}, \tag{3.11}$$

with $M_i^{(\infty)} = (W_1 + \cdots + W_i)^{-\frac{1}{\alpha}}$ and the W_i's are i.i.d. exponential random variables of parameter one and $(Y_i^{(\infty)})_{i\in\mathbb{N}}$ is a sequence of i.i.d. uniform random variables on $[0,1]$. We consider the continuum version of the variational equation (3.9)

$$\hat{I}_{\hat{\beta},\infty} = \arg\max_{I\in\mathbf{X}} \left\{\hat{\beta}\hat{\sigma}_\infty(I) - \mathrm{c}\, E(I)\right\}, \tag{3.12}$$

where $\hat{\sigma}_\infty(I) := \sum_{i\in\mathbb{N}} M_i^{(\infty)} \mathbb{1}(Y_i^{(\infty)} \in I)$ is the continuum energy. Then we have the following

Theorem 3.3. *Let $(\beta_N)_N$ be as in* (3.8)*. For any $\hat{\beta} > 0$ let $\hat{I}_{\hat{\beta},\infty} \in \mathbf{X}$ be as in* (3.12)*, then*

$$I_{\beta_N,N} \xrightarrow{(\mathrm{d})} \hat{I}_{\hat{\beta},\infty} \tag{3.13}$$

on $(\mathbf{X}, d_H)$*.*

As consequence if we look at $\mathrm{P}^{\omega}_{\beta_N,h,N}$ as a random probability on $\mathbf{X}$, i.e. as a random variable which takes values in $\mathcal{M}_1(\mathbf{X}, d_H)$, the space of the probability measures on $\mathbf{X}$, then Theorems 3.2 and 3.3 imply that it converges in law to the δ-measure concentrated on the limit set $\hat{I}_{\hat{\beta},\infty}$.

Theorem 3.4. *Let $(\beta_N)_N$ be as in (3.8). Then for any $h, \hat{\beta} \in (0,\infty)$,*

$$\mathrm{P}^{\omega}_{\beta_N,h,N} \xrightarrow{(\mathrm{d})} \delta_{\hat{I}_{\hat{\beta},\infty}} \tag{3.14}$$

on $\mathcal{M}_1(\mathbf{X}, d_H)$ equipped with the weak topology.

This concludes the results on the convergence of the random set $\tau/N \cap [0,1]$. By definition of Hausdorff distance to say when $\tau/N \cap [0,1]$ has a trivial limit or not it is sufficient to study when the limit set $\hat{I}_{\hat{\beta},\infty}$ is given by $\{0,1\}$ or not. For this purpose we define the *random threshold* $\hat{\beta}_c$ as

$$\hat{\beta}_c = \inf\{\hat{\beta} : \hat{I}_{\hat{\beta},\infty} \not\equiv \{0,1\}\}. \tag{3.15}$$

Denoting by $\mathbb{P}$ the law of the continuum disorder, we have that

(1) If $\hat{\beta} < \hat{\beta}_c$ then $\hat{I}_{\hat{\beta},\infty} \equiv \{0,1\}$, $\mathbb{P}$-a.s.

(2) If $\hat{\beta} > \hat{\beta}_c$ then $\hat{I}_{\hat{\beta},\infty} \not\equiv \{0,1\}$, $\mathbb{P}$-a.s.

Exactly as in the directed polymer in a random environment with heavy tails, also in this case we conjecture that for any $\hat{\beta}$, the set $\hat{I}_{\hat{\beta},\infty}$ is given by a finite number of points, but we do not have any proof. Let us stress that in principle the difficulty of proving such conjecture is the same in both models. On the other hand, the structure of $\hat{\beta}_c$, figure 3.2, is described by the following theorem

Theorem 3.5. *For any choice of $\alpha, \gamma \in (0,1)$ we have that $\hat{\beta}_c > 0$ $\mathbb{P}$-a.s., where α is the disorder exponent, while γ is the renewal exponent.*

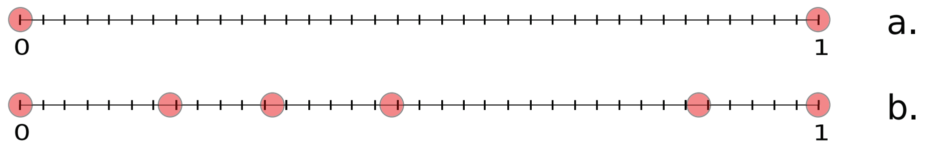

Fig. 3.2: In red we have marked the points of $\hat{I}_{\hat{\beta},\infty}$. In (a) we have $\hat{\beta} < \hat{\beta}_c$ and $\hat{I}_{\hat{\beta},\infty} \equiv \{0,1\}$,$\mathbb{P}$-a.s., while in (b) we have $\hat{\beta} > \hat{\beta}_c$ and $\hat{I}_{\hat{\beta},\infty} \not\equiv \{0,1\}$, $\mathbb{P}$-a.s.

We conclude this section by improving Theorem 2.10 about the structure of β_c, the critical threshold of the directed polymer in a random environment with heavy tails. Precisely

Theorem 3.6. *Let $\mathbb{P}_\infty$ be the law of the continuum environment, then*

(1) *For any $\alpha \in (0, \frac{1}{2})$, $\beta_c > 0$, $\mathbb{P}_\infty$-a.s.*

(2) *For any $\alpha \in [\frac{1}{2}, 2)$, $\beta_c = 0$, $\mathbb{P}_\infty$-a.s.*

3.2 Universality for the pinning model in the weak coupling regime

In this section we aim to make formal the discussion in Sections 2.3.1 and 2.3.3. In particular our main goal is to prove the conjecture (2.54). We prove such results in Chapter 5 and, Theorem 3.7, in Chapter 6.

In the sequel we consider the disordered pinning models introduced in Section 2.3 and, inspired by Caravenna, Sun and Zygouras [23, 24], we prove sharp estimates for partition functions, free energy and critical curve in the weak coupling regime: we show that the free energy and critical point of discrete pinning models, suitably rescaled, converge to the analogous quantities of related continuum models. This is obtained by a suitable coarse-graining procedure, which generalizes and refines [16, 22].

In the following we are going to give our precise assumptions on the model and then we explain our results. We will also give a precise description of our coarse-graining procedure which has an independent interest of the main results.

3.2.1 General assumptions

According to the assumptions made in Section 2.3.3, through this section we consider a non-terminating renewal process, cf. Sections 2.2 and 2.5.3 with *heavy tail*, and a disorder, cf. Section 2.3, with finite exponential moments, i.e.,

A1. The renewal process τ satisfies

$$\begin{aligned} &\mathrm{P}(\tau_1 < \infty) = 1, \\ &K(n) := \mathrm{P}(\tau_1 = n) \sim \frac{L(n)}{n^{1+\alpha}}, \quad n \to \infty; \quad \alpha \in (\frac{1}{2}, 1) \\ &K(n) > 0\,, \forall\, n \in \mathbb{N}, \end{aligned} \tag{3.16}$$

where $L(n)$ is a slowly varying function [14]. We have to strengthen our assumptions on such renewal process, in particular we assume that the convergence of its *renewal function* $u(n) := \mathrm{P}(N \in \tau)$ takes place at a not too slow rate, i.e. at least a power law of $\frac{\ell}{n}$, as in [24, eq. (1.7)]:

$$\exists\, C, n_0 \in (0,\infty);\ \varepsilon,\delta \in (0,1]: \quad \left|\frac{u(n+\ell)}{u(n)} - 1\right| \le C\left(\frac{\ell}{n}\right)^{\delta}, \quad \forall n \ge n_0,\ 0 \le \ell \le \varepsilon n. \tag{3.17}$$

Let us stress that this is the same assumption needed in Theorem 2.8.

A2. The disorder variables have locally finite exponential moments:

$$\exists \beta_0 > 0:\ \Lambda(\beta) := \log \mathbb{E}(e^{\beta\omega_1}) < \infty,\ \forall \beta \in (-\beta_0, \beta_0), \qquad \mathbb{E}(\omega_1) = 0, \qquad \mathbb{V}(\omega_1) = 1\,, \tag{3.18}$$

and it satisfies the following concentration inequality:

$$\begin{aligned} &\exists \gamma \ge 1,\ C_1, C_2 \in (0,\infty):\ \text{for all } n \in \mathbb{N} \text{ and for all } f:\mathbb{R}^n \to \mathbb{R} \text{ convex and 1-Lipschitz} \\ &\mathbb{P}\Big(\big|f(\omega_1,\dots,\omega_n) - M_f\big| \ge t\Big) \le C_1 \exp\Big(-\frac{t^\gamma}{C_2}\Big), \end{aligned} \tag{3.19}$$

where 1-Lipschitz means $|f(x) - f(y)| \le |x - y|$ for all $x, y \in \mathbb{R}^n$, with $|\cdot|$ the usual Euclidean norm, and M_f denotes a median of $f(\omega_1,\dots,\omega_n)$. (One can equivalently take M_f to be the mean $\mathbb{E}[f(\omega_1,\dots,\omega_n)]$ just by changing the constants C_1, C_2, cf. [63, Proposition 1.8].)

It is known that (3.19) holds under fairly general assumptions, namely:

- ($\gamma = 2$) if ω_1 is bounded, i.e. $\mathbb{P}(|\omega_1| \le a) = 1$ for some $a \in (0,\infty)$, cf. [63, Corollary 4.10];
- ($\gamma = 2$) if the law of ω_1 satisfies a log-Sobolev inequality, in particular if ω_1 is Gaussian, cf. [63, Theorems 5.3 and Corollary 5.7]; more generally, if the law of ω_1 is absolutely continuous with density $\exp(-U - V)$, where U is uniformly strictly convex (i.e. $U(x) - cx^2$ is convex, for some $c > 0$) and V is bounded, cf. [63, Theorems 5.2 and Proposition 5.5];
- ($\gamma \in (1,2)$) if the law of ω_1 is absolutely continuous with density given by $c_\gamma\, e^{-|x|^\gamma}$ (see Propositions 4.18 and 4.19 in [63] and the following considerations).

3.2.2 Main results

Our principal aim is to find the sharp asymptotics of the critical point $h_c(\beta)$, introduced in (2.31), by proving conjecture (2.54): we prove that for any $\alpha \in (1/2, 1)$ there exists a constant $m_\alpha \in (0,\infty)$ and a slowly varying function $\tilde{L}_\alpha$, uniquely defined by L and α, cf. Remark 5.2 below, such that

$$\lim_{\beta \to 0} \frac{h_c(\beta)}{\tilde{L}_\alpha(\frac{1}{\beta})\,\beta^{\frac{2\alpha}{2\alpha-1}}} = m_\alpha. \tag{3.20}$$

Our starting point is the continuum partition function $\mathbf{Z}^W_{\hat\beta,\hat h}(t)$ defined in Theorem 2.8. In analogy with the discrete model it is natural to define a *continuum free energy* $\mathbf{F}^\alpha(\hat\beta,\hat h)$ in terms of it. Our first result ensures the existence of such a quantity and gives its scaling properties.

Theorem 3.7 (Continuum free energy). *For all $\alpha \in (\frac{1}{2},1)$, $\hat\beta > 0$, $\hat h \in \mathbb{R}$ the following limit exists and is finite:*

$$\mathbf{F}^\alpha(\hat\beta,\hat h) := \lim_{t\to\infty} \frac{1}{t} \log \mathbf{Z}^W_{\hat\beta,\hat h}(t) = \lim_{t\to\infty} \frac{1}{t} \mathbb{E}\Big[\log \mathbf{Z}^W_{\hat\beta,\hat h}(t)\Big], \qquad \mathbb{P}\text{-a.s. and in } L^1. \tag{3.21}$$

The function $\mathbf{F}^\alpha(\hat\beta,\hat h)$ is non-negative: $\mathbf{F}^\alpha(\hat\beta,\hat h) \geq 0$ for all $\hat\beta > 0$, $\hat h \in \mathbb{R}$. Furthermore, it is a convex function of $\hat h$, for fixed $\hat\beta$, and satisfies the following scaling relation:

$$\mathbf{F}^\alpha(c^{\alpha-\frac{1}{2}}\hat\beta, c^\alpha \hat h) = c\,\mathbf{F}^\alpha(\hat\beta,\hat h), \qquad \forall \hat\beta > 0,\ \hat h \in \mathbb{R},\ c \in (0,\infty). \tag{3.22}$$

Note that, by (3.22),

$$\mathbf{F}^\alpha(\hat\beta,\hat h) = \mathbf{F}^\alpha\Big(1, \frac{\hat h}{\hat\beta^{\frac{2\alpha}{2\alpha-1}}}\Big)\hat\beta^{\frac{2}{2\alpha-1}}, \qquad \text{hence} \qquad \mathbf{h}^\alpha_c(\hat\beta) = \mathbf{h}^\alpha_c(1)\hat\beta^{\frac{2\alpha}{2\alpha-1}}, \tag{3.23}$$

where $\mathbf{h}^\alpha_c(\hat\beta)$ is the continuum critical point defined in (2.52).

The following theorem, which is our main result, shows that (3.20) is indeed justified. We actually prove a stronger relation, which also yields the precise asymptotic behavior of the critical curve.

Theorem 3.8 (Interchanging the limits). *Let $\mathrm{F}(\beta,h)$ be the free energy of a disordered pinning model* (2.27)*, where the renewal process τ satisfies* (3.16)-(3.17) *for some $\alpha \in (\frac{1}{2},1)$ and the disorder ω satisfies* (3.18)-(3.19)*. For all $\hat\beta > 0$, $\hat h \in \mathbb{R}$ and $\eta > 0$ there exists $\varepsilon_0 > 0$ such that*

$$\mathbf{F}^\alpha\left(\hat\beta, \hat h - \eta\right) \leq \frac{\mathrm{F}(\hat\beta\,\varepsilon^{\alpha-\frac{1}{2}}L(\frac{1}{\varepsilon}),\ \hat h\,\varepsilon^\alpha L(\frac{1}{\varepsilon}))}{\varepsilon} \leq \mathbf{F}^\alpha\left(\hat\beta, \hat h + \eta\right), \qquad \forall \varepsilon \in (0,\varepsilon_0). \tag{3.24}$$

As a consequence, relation (2.51) *holds, and furthermore*

$$\lim_{\beta\to 0} \frac{h_c(\beta)}{\tilde L_\alpha(\frac{1}{\beta})\beta^{\frac{2\alpha}{2\alpha-1}}} = \mathbf{h}^\alpha_c(1), \tag{3.25}$$

where $\tilde L_\alpha$ is the slowly function appearing in (3.20) *and $\mathbf{h}^\alpha_c(\hat\beta)$ is the continuum critical point defined in* (2.52).

Remark 3.9. Note that relation (3.25) follows immediately by (3.24), sending $\eta \to 0$, because $\hat h \mapsto \mathbf{F}^\alpha(\hat\beta,\hat h)$ is continuous (by convexity, cf. Theorem 3.7).

3.2.3 Further results

Our results on the free energy and critical curve are based on a comparison of discrete and continuum partition function, through a coarse-graining procedure. Some of the intermediate results are of independent interest and are presented here.

Let us consider the "free" partition function $\mathrm{Z}^\omega_{\beta_N,h_N}(N)$, and the conditioned one $\mathrm{Z}^{\omega,\mathrm{c}}_{\beta_N,h_N}(M,N)$, defined in (2.28) and (2.29) respectively. We recall, cf. Theorem 2.8, that for any fixed $\hat h$ and $\hat\beta$, by linearly interpolating such partition functions for $Ns, Nt \notin \mathbb{N}_0$, one has convergence in distribution to $\mathbf{Z}^W_{\hat\beta,\hat h}(t)$ and $\mathbf{Z}^{W,\mathrm{c}}_{\hat\beta,\hat h}(s,t)$ the continuum (free and conditioned) partition function, respectively, in the space of continuous functions $t \in [0,\infty)$ and $(s,t) \in [0,\infty)^2_\leq := \{(s,t) \in [0,\infty)^2 \mid s \leq t\}$ equipped with the topology of uniform convergence on compact sets.

We can strengthen this result, by showing that the convergence is locally uniform also in the variable $\hat{h} \in \mathbb{R}$. We formulate this fact through the existence of a suitable coupling.

Theorem 3.10 (Uniformity in $\hat{h}$). *Assume* (3.16)-(3.17)*, for some* $\alpha \in (\frac{1}{2}, 1)$*, and* (3.18)*. For all* $\hat{\beta} > 0$*, there is a coupling of discrete and continuum partition functions such that the convergence of* $\left(\mathrm{Z}^{\omega}_{\beta_N,h_N}(Nt)\right)_{t\geq 0}$*, resp.* $\left(\mathrm{Z}^{\omega,\mathsf{c}}_{\beta_N,h_N}(Ns,Nt)\right)_{0\leq s\leq t}$*, holds* $\mathbb{P}(\mathrm{d}\omega,\mathrm{d}W)$*-a.s. uniformly in any compact set of values of* $(t,\hat{h})$*, resp. of* $(s,t,\hat{h})$*.*

Since $h \mapsto \log \mathrm{Z}^{\omega}_{\beta,h}$ and $h \mapsto \log \mathrm{Z}^{\omega,\mathsf{c}}_{\beta,h}$ are convex functions and convexity is preserved under pointwise convergence, we obtain the following:

Proposition 3.11. *For all* $\alpha \in (\frac{1}{2}, 1)$ *and* $\hat{\beta} > 0$*, the processes* $\mathbf{Z}^{W}_{\hat{\beta},\hat{h}}(t)$ *and* $\mathbf{Z}^{W,\mathsf{c}}_{\hat{\beta},\hat{h}}(s,t)$ *admit a version which is continuous in* $(t,\hat{h})$ *and in* $(s,t,\hat{h})$*, respectively, and log-convex in* $\hat{h}$*.*

We conclude with some important estimates, bounding (positive and negative) moments of the partition functions and providing a deviation inequality.

Proposition 3.12. *Assume* (3.16)-(3.17)*, for some* $\alpha \in (\frac{1}{2}, 1)$*, and* (3.18)*. Fix* $\hat{\beta} > 0$*,* $\hat{h} \in \mathbb{R}$*. For all* $T > 0$ *and* $p \in [0,\infty)$*, there exists a constant* $C_{p,T} < \infty$ *such that*

$$\mathbb{E}\left[\sup_{0\leq s\leq t\leq T} \mathrm{Z}^{\omega,\mathsf{c}}_{\beta_N,h_N}(Ns,Nt)^p\right] \leq C_{p,T}\,, \qquad \forall N \in \mathbb{N}\,. \tag{3.26}$$

Assuming also (3.19)*, relation* (3.26) *holds also for every* $p \in (-\infty, 0]$*, and furthermore*

$$\sup_{0\leq s\leq t\leq T} \mathbb{P}\Big(\log \mathrm{Z}^{\omega,\mathsf{c}}_{\beta_N,h_N}(Ns,Nt) \leq -x\Big) \leq A_T \exp\left(-\frac{x^{\gamma}}{B_T}\right), \qquad \forall x \geq 0,\ \forall N \in \mathbb{N}\,, \tag{3.27}$$

for suitable finite constants A_T*,* B_T*. Finally, relations* (3.26)*,* (3.27) *hold also for the free partition function* $\mathrm{Z}^{\omega}_{\beta_N,h_N}(Nt)$ *(replacing* $\sup_{0\leq s\leq t\leq T}$ *with* $\sup_{0\leq t\leq T}$*).*

For relation (3.27) we use the concentration assumptions (3.19) on the disorder. However, since $\log \mathrm{Z}^{\omega,\mathsf{c}}_{\beta_N,h_N}$ is not a uniformly (over $N \in \mathbb{N}$) Lipschitz function of ω, some work is needed.

Finally, since the convergences in distribution of $\left(\mathrm{Z}^{\omega}_{\beta_N,h_N}(Nt)\right)_{t\geq 0}$ and $\left(\mathrm{Z}^{\omega,\mathsf{c}}_{\beta_N,h_N}(Ns,Nt)\right)_{0\leq s\leq t}$ holds in the space of continuous functions, we can easily deduce analogues of (3.26), (3.27) for the continuum partition functions, leading to our last result.

Corollary 3.13. *Fix* $\alpha \in (\frac{1}{2}, 1)$*,* $\hat{\beta} > 0$*,* $\hat{h} \in \mathbb{R}$*. For all* $T > 0$ *and* $p \in \mathbb{R}$ *there exist finite constants* A_T*,* B_T*,* $C_{p,T}$ *(depending also on* $\alpha, \hat{\beta}, \hat{h}$*) such that*

$$\mathbb{E}\left[\sup_{0\leq s\leq t\leq T} \mathbf{Z}^{W,\mathsf{c}}_{\hat{\beta},\hat{h}}(Ns,Nt)^p\right] \leq C_{p,T}\,, \tag{3.28}$$

$$\sup_{0\leq s\leq t\leq T} \mathbb{P}\Big(\log \mathbf{Z}^{W,\mathsf{c}}_{\hat{\beta},\hat{h}}(Ns,Nt) \leq -x\Big) \leq A_T \exp\left(-\frac{x^{\gamma}}{B_T}\right), \qquad \forall x \geq 0\,. \tag{3.29}$$

The same relations hold for the free partition function $\mathbf{Z}^{W}_{\hat{\beta},\hat{h}}(t)$ *(replacing* $\sup_{0\leq s\leq t\leq T}$ *with* $\sup_{0\leq t\leq T}$*).*

3.3 Conclusion and perspectives

The work developed in this thesis has contributed to better understand the critical properties of the pinning model. In this section we want to summarize briefly the importance of the results obtained and give a perspective on the open problems connected with the thesis.

The first research subject of the thesis, the pinning model with heavy-tailed disordered, cf. Section 3.1, represents the first analysis of such model with this particular choice of the disorder. There are several open questions regarding mainly, but not only, the comprehension of the model with different choices of renewal process:

- ($\gamma \geq 1$) The condition $\gamma \geq 1$ implies that the Entropy function $E(I)$, cf. (3.7), is non-increasing (strictly non-increasing if $\gamma > 1$) with respect to the inclusion of sets in $\mathbf{X}^{(N)}$, cf. (3.2). It turns out that for any fixed $\beta > 0$ and $N \in \mathbb{N}$, the solution of (4.13) is $I_{\beta,N} = \{0, 1/N, \cdots, 1\}$. Therefore, whenever $N \to \infty$, the limit set is given by the interval $[0, 1]$, independently of our choice of β_N. We conjecture that τ/N converges to the whole segment $[0, 1]$.

- ($\gamma = 0$) The case $\gamma = 0$ corresponds to consider a renewal process with polynomial tail, that is $K(n) := \mathrm{P}(\tau_1 = n) \sim L(n)n^{-\rho}$, with $\rho > 1$, cf. (4.4). In this case we conjecture that the correct rescaling is given by $\beta = \beta_N \sim N^{-1/\alpha} \log N$ and the limit measure for the the sequence $\mathrm{P}^{\omega}_{\beta_N,h,N}(\cdot)$ is given by a more complicated structure than the δ-measure of a single set. This would mean that we do not have concentration around a single favorable set.

- An interesting open problem is given by the structure of $\hat{I}_{\hat{\beta},\infty}$, cf. (3.12). In Theorem 3.5 we have proven that if $\hat{\beta}$ is small enough, then $\hat{I}_{\hat{\beta},\infty} \equiv \{0, 1\}$ a.s., otherwise, if $\hat{\beta}$ is large, $\{0, 1\} \subsetneq \hat{I}_{\hat{\beta},\infty}$. We conjecture that for any finite $\hat{\beta} > 0$ it is given by a finite number of points.

  Moreover, whenever $h \geq 0$, the renewal process under the pinning model measure converges to the whole segment $[0, 1]$, cf. Section 3.1.4 and Proposition 2.16. This means that for any fixed h the random set $\hat{I}_{\hat{\beta},\infty}$ would present a simple structure: a finite number of points if $h < 0$, an interval if $h \geq 0$. Therefore an interesting problem regards the existence of a suitable rescaling of $h_N \uparrow 0$ as $N \to \infty$, with $h_N < 0$, such that the renewal process under the pinning model measure converges to a non-trivial limit set.

Another ongoing research programme regards the directed polymer in a heavy tailed environment. Indeed in [7] the authors studied the case in which the distribution function of the disorder decays polynomially with exponent $\alpha < d$, where d is the dimension of the ambient space. In dimension $d = 1 + 1$ they showed that the fluctuations of the polymer are of order N, where N is the polymer length. It is conjectured that if $\alpha \geq 5$, then the fluctuation exponent of the trajectory is the same as in the Gaussian case [2]. The intermediate case of $2 < \alpha < 5$ is still an open problem. One conjectures that the fluctuations of the model are tuned by an exponent μ which interpolates between the case $\alpha \leq 2$ – for which $\mu = 1$ [7] – and $\alpha \geq 5$, where it is conjectured, and for some specific disorder laws proven [75], to be $\mu = 2/3$. Such conjecture is supported by also non-matching upper and lower bounds given for certain special models [68, 70, 81, 82]. Let us underline that this problem is deeply connected with the analogous problem for the last-passage percolation with heavy tailed weights [66, 50].

The second research work, the universality of the pinning model in the weak disorder regime, cf. Section 3.2, represents the main result of this thesis. In this work we have solved a challenging open problem concerning the critical properties of the pinning model when the disorder is small. To be more precise, we have proven that the behavior of the pinning model in the weak disorder regime is universal and the critical point, suitably rescaled, converges to the related quantity of a continuum model. The result is obtained by using a coarse-graining procedure, which generalizes the technique developed for the copolymer model [16, 22].

In [1, 2] the authors have introduce the concept of *intermediate disorder regime* for the directed polymer model in random environment, cf. Section 2.4. It corresponds to scale the parameter β with N, the length of the polymer. In particular if $\beta_N = \beta N^{-\frac{1}{4}}$ the partition function of the (discrete) model has a continuum limit. In [23] this result has been generalized to a larger class of random walks.

Such analogy with the disordered pinning model suggests that the critical properties of the model can converge to a universal limit. The conjecture is that the free energy of (discrete) directed polymer, suitably rescaled, converges to the analogous quantity of the continuum model. To be more precise, let us can consider the partition function of the directed polymer model, $Z^{\omega}_{N,\beta}$, and the free energy

$$F(\beta) = \lim_{N\to\infty} \frac{1}{N}\mathbb{E}\left[\log Z^{\omega}_{N,\beta}\right].$$

In [1, 2] it was proven that if we rescale $\beta = \hat{\beta} N^{-\frac{1}{4}}$, then the partition function admits a continuum limit: for any fixed $t > 0$ one has

$$\lim_{N\to\infty} Z^{\omega}_{Nt,\beta} \overset{(d)}{=} \mathbf{Z}^{W}_{t,\hat{\beta}}.$$

Assuming that the continuum free energy

$$\mathbf{F}(\hat{\beta}) = \lim_{N\to\infty} \frac{1}{N}\mathbb{E}\left[\log \mathbf{Z}^{W}_{t,\hat{\beta}}\right]$$

exists. It is conjectured in [23] that

$$F(\beta) \underset{\beta\to 0}{\sim} \mathbf{F}(1)\beta^4.$$

This conjecture is supported by the (non-matching) upper and lower bounds studied in [61]. A possible strategy to prove such conjecture is given by a coarse-graining procedure analogous to the one developed for the pinning model. Let us stress again that the interest for these results goes beyond the model itself: such result would show that many popular models of directed random polymers (pinning, copolymer and directed polymer in random environment) display universal feature described by a suitable continuum limit model. Moreover the technique used would always be the same — the coarse-graining decomposition — and this suggests a general approach to study this kind of problem. We would be very interested in showing that the same approach works for other kind of statistical mechanics models for which a continuum limit exists. One of such models is represented by the *(disordered) Ising model*, for which the existence of a continuum limit has been proved in [23, 27, 18, 19, 20].

Pinning model with heavy-tailed disorder

4

In this section we prove the results we obtained for a disordered pinning model with *heavy-tailed disorder*, with exponent $\alpha \in (0,1)$ discussed in Chapter 3.1.

In particular we consider a disordered pinning model, which describes the behavior of a Markov chain interacting with a distinguished state. The interaction depends on an external source of randomness, called disorder, which can attract or repel the Markov chain path, and is tuned by a parameter β. Inspired by [7, 50], we focus on the case when the disorder is heavy-tailed, with exponent $\alpha \in (0,1)$, while the return times of the Markov chain have a stretched-exponential distribution, with exponent $\gamma \in (0,1)$. We prove that the set of times at which the Markov chain visits the distinguished state, suitably rescaled, converges in distribution to a limit set, which depends only on the disorder and on the interplay of the parameters α, γ, β. We also show that there exists a random threshold of β below which the limit set is trivial. As a byproduct of our techniques, we improve and complete a result of A.Auffinger and O.Louidor [7, proposition 2.5] on the directed polymer in a random environment with heavy tailed disorder.

The article [78] has been taken from the content of this chapter.

4.1 Set-up and Results

The pinning model can be defined as a random perturbation of a random walk or, more generally, of a Markov chain called S. In this model we modify the law of the Markov chain by weighing randomly the probability of a given trajectory up to time N. Each time S touches a distinguished state, called 0, before N, say at time n, we give a reward or a penalty to this contact by assigning an exponential weight $\exp(\beta\omega_n - h)$, where $\beta \in \mathbb{R}_+ := (0, \infty)$, $h \in \mathbb{R}$ and $(\omega = (\omega_n)_{n\in\mathbb{N}}, \mathbb{P})$ is an independent random sequence called disorder. The precise definition of the model is given below.

In this model we perturb S only when it takes value 0, therefore it is convenient to work with its zero level set. For this purpose we consider a renewal process $(\tau = (\tau_n)_{n\in\mathbb{N}}, \mathrm{P})$, that is an $\mathbb{N}_0$-valued random process such that $\tau_0 = 0$ and $(\tau_j - \tau_{j-1})_{j\in\mathbb{N}}$ is an i.i.d. sequence. This type of random process can be thought of as a random subset of $\mathbb{N}_0$, in particular if $S_0 = 0$, then by setting $\tau_0 = 0$ and $\tau_j = \inf\{k > \tau_{j-1} : S_k = 0\}$, for $j > 0$, we recover the zero level set of the Markov chain S. From this point of view the notation $\{n \in \tau\}$ means that there exists $j \in \mathbb{N}$ such that $\tau_j = n$. We refer to [6, 44] for more details about the theory of the renewal processes.

In the literature, e.g. [31, 45, 44], typically the law of τ_1, the inter-arrival law of the renewal process, has a polynomial tail and the disorder has finite exponential moments. In this chapter we study the case in which the disorder has polynomial tails, in analogy with the articles [7] and [50]. To get interesting results we work with a renewal process where the law of τ_1 is stretched-exponential (cf. Assumptions 4.2).

4.1.1 The Pinning Model

In this chapter we want to understand the behavior of $\tau/N \cap [0,1] = \{\tau_j/N : \tau_j \leq N\}$, the rescaled renewal process up to time N,when N gets large.

We denote by P_N the law of $\tau/N \cap [0,1]$, which turns out to be a probability measure on the space of all subsets of $\{0, 1/N, \cdots, 1\}$. On this space, for $\beta, h \in \mathbb{R}$ we define the *pinning model* $\mathrm{P}^{\omega}_{\beta,h,N}$ as a probability measure defined by the following Radon-Nikodym derivative

$$\frac{\mathrm{dP}^{\omega}_{\beta,h,N}}{\mathrm{dP}_N}(I) = \frac{1}{\mathrm{Z}^{\omega}_{\beta,h,N}} \exp\left(\sum_{n=1}^{N-1}(\beta\omega_n - h)\mathbb{1}(n/N \in I)\right)\mathbb{1}(1 \in I), \tag{4.1}$$

where $\mathrm{Z}^{\omega}_{\beta,h,N}$ is a normalization constant, called partition function, that makes $\mathrm{P}^{\omega}_{\beta,h,N}$ a probability. Let us stress that a realization of $\tau/N \cap [0,1]$ has non-zero probability only if its last point is equal to 1. This is due to the presence of the term $\mathbb{1}(1 \in I)$ in (4.1). In such a way the pinning model is a *random* probability measure on the space $\mathbf{X}$ of all closed subsets of $[0,1]$ which contain both 0 and 1

$$\mathbf{X} = \{I \subset [0,1] : I \text{ is closed and } 0, 1 \in I\} \tag{4.2}$$

with support given by $\mathbf{X}^{(N)}$, the set of all subsets of $\{0, 1/N, \cdots, 1\}$ which contains both 0 and 1.

The pinning model $\mathrm{P}^{\omega}_{\beta,h,N}$ is a *random* probability measure, in the sense that it depends on a parameter ω, called disorder, which is a quenched realization of a random sequence. Therefore in the pinning model we have two (independent) sources of randomness: the renewal process (τ, P) and the disorder $(\omega, \mathbb{P})$. To complete the definition we thus need to specify our assumptions about the disorder and the renewal process.

Assumption 4.1. *We assume that the disorder ω is an i.i.d. sequence of random variables whose tail is regularly varying with index $\alpha \in (0,1)$, namely*

$$\mathbb{P}(\omega_1 > t) \sim L_0(t)t^{-\alpha}, \quad t \to \infty, \tag{4.3}$$

where $\alpha \in (0,1)$ and $L_0(\cdot)$ is a slowly varying function, cf. [14]. Moreover we assume that the law of ω_1 has no atom and it is supported in $(0,\infty)$, i.e. ω_1 is a positive random variables. The reference example to consider is given by the Pareto Distribution.

Assumption 4.2. *Given a renewal process, we denote the law of its first point τ_1 by $K(n) := \mathrm{P}(\tau_1 = n)$, which characterizes completely the process. Throughout the chapter we consider a non-terminating renewal process τ, i.e., $\sum_{n\in\mathbb{N}} K(n) = 1$, which satisfies the two following assumptions*

(1) *Subexponential, cf. Appendix 4.A:*

$$\forall\, k > 0,\ \lim_{n\to\infty} K(n+k)/K(n) = 1 \quad \textit{and} \quad \lim_{n\to\infty} K^{*(2)}(n)/K(n) = 2,$$

(2) *Stretched-exponential*

$$\exists\, \gamma \in (0,1),\ \mathsf{c} > 0 : \quad \lim_{n\to\infty} \log K(n)/n^{\gamma} = -\mathsf{c}$$

Remark 4.3. Roughly speaking, up to local regularity assumptions (subexponentiality), we take $K(n) \cong e^{-\mathsf{c}n^{\gamma}}$. More precisely these conditions are satisfied if

$$K(n) \sim \frac{L(n)}{n^{\rho}} e^{-\mathsf{c}n^{\gamma}}, \quad n \to \infty, \tag{4.4}$$

with $\rho \in \mathbb{R}$ and $L(\cdot)$ a slowly varying function, cf. Section 4.A.

4.1.2 Main Results

The aim of this chapter is to study the behavior of $\tau/N \cap [0,1]$ under the probability $\mathrm{P}^{\omega}_{\beta,h,N}$, when N gets large. To have a non trivial behavior we need to fix $h > 0$ (which is actually equivalent to set $h = 0$ in (4.1) and consider a terminating renewal process, cf. Section 4.4.1) and send β to 0 as $N \to \infty$. If β goes to 0 too slowly (or if it does not go to 0 at all), then $\tau/N \cap [0,1]$ will always converge to the whole $[0, 1]$, if it goes too fast, it will converge to $\{0, 1\}$. The interesting regime is the following:

$$\beta_N \sim \hat{\beta} N^{\gamma - \frac{1}{\alpha}} \ell(N), \quad N \to \infty, \tag{4.5}$$

with ℓ a particular slowly varying function defined by L_0 in (4.3). Under such rescaling of β and such choice of $h > 0$ we prove the existence of a random threshold $\hat{\beta}_c$: if $\hat{\beta} < \hat{\beta}_c$ then $\tau/N \cap [0, 1]$ converges to $\{0, 1\}$, while if $\hat{\beta} > \hat{\beta}_c$ then its limit has at least one point in $(0, 1)$.

To prove these facts we proceed by steps. In the first one we show that there exists a random set around which $\tau/N \cap [0, 1]$ is concentrated with respect to the Hausdorff distance: given two non-empty sets $A, B \subset [0, 1]$

$$d_H(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\}, \tag{4.6}$$

where $d(z, C) = \inf_{c \in C} |z - c|$ is the usual distance between a point and a set.

Theorem 4.4. *Let $(\beta_N)_N$ be as in* (4.5). *Then for any $N \in \mathbb{N}$, $\hat{\beta} > 0$ there exists a random set $I_{\beta_N,N}$ (i.e. a $\mathbf{X}$-valued random variable) such that for any $\delta, h > 0$ one has that $\mathrm{P}^{\omega}_{\beta_N,h,N} \left(d_H(I, I_{\beta_N,N}) > \delta \right)$ converges to 0 as $N \to \infty$ in probability (with respect to the disorder ω). More precisely for any $\varepsilon > 0$ there exists $\nu = \nu(\varepsilon, \delta)$ and $\hat{N}$ such that for all $N > \hat{N}$*

$$\mathbb{P} \left(\mathrm{P}^{\omega}_{\beta_N,h,N} \left(d_H(I, I_{\beta_N,N}) > \delta \right) < e^{-\nu N^{\gamma}} \right) > 1 - \varepsilon. \tag{4.7}$$

The second step regards the convergence in law of $I_{\beta_N,N}$.

Theorem 4.5. *Let $(\beta_N)_N$ be as in* (4.5). *Then for any $\hat{\beta} > 0$ there exists a random closed subset $\hat{I}_{\hat{\beta},\infty} \in \mathbf{X}$ (i.e. a $\mathbf{X}$-valued random variable), which depends on a suitable continuum disorder (defined in section 4.2.1), such that*

$$I_{\beta_N,N} \xrightarrow{(d)} \hat{I}_{\hat{\beta},\infty}, \quad N \to \infty \tag{4.8}$$

on $(\mathbf{X}, d_H)$.

As a consequence of these Theorems, if we look at $\mathrm{P}^{\omega}_{\beta_N,h,N}$ as a random probability on $\mathbf{X}$, i.e. as a random variable in $\mathcal{M}_1(\mathbf{X}, d_H)$, the space of the probability measures on $\mathbf{X}$, then Theorems 4.4 and 4.5 imply that it converges in law to the δ-measure concentrated on the limit set $\hat{I}_{\hat{\beta},\infty}$.

Theorem 4.6. *Let $(\beta_N)_N$ be as in* (4.5). *Then for any $h, \hat{\beta} \in (0, \infty)$,*

$$\mathrm{P}^{\omega}_{\beta_N,h,N} \xrightarrow{(d)} \delta_{\hat{I}_{\hat{\beta},\infty}}, \quad N \to \infty \tag{4.9}$$

on $\mathcal{M}_1(\mathbf{X}, d_H)$ equipped with the weak topology.

This concludes our results about the convergence of the random set $\tau/N \cap [0, 1]$, now we want to discuss the structure of its limit. We prove that there exists a critical point $\hat{\beta}_c$ such that, if $\beta < \hat{\beta}_c$, then $\tau/N \cap [0, 1]$ has a trivial limit, given by $\{0, 1\}$. Otherwise, if $\beta > \hat{\beta}_c$, then the limit sets has points in $(0, 1)$.

We define the random threshold $\hat{\beta}_c$ as

$$\hat{\beta}_c = \inf\{\hat{\beta} : \hat{I}_{\hat{\beta},\infty} \not\equiv \{0, 1\}\}. \tag{4.10}$$

Denoting by $\mathbb{P}$ the law of the continuum disorder, by a monotonicity argument (cf. Section 4.5) we have that

(1) If $\hat{\beta} < \hat{\beta}_c$, then $\hat{I}_{\hat{\beta},\infty} \equiv \{0, 1\}$, $\mathbb{P}$-a.s.

(2) If $\hat{\beta} > \hat{\beta}_c$, then $\hat{I}_{\hat{\beta},\infty} \not\equiv \{0,1\}$, $\mathbb{P}$-a.s.

Moreover the structure of $\hat{\beta}_c$ is described by the following result

Theorem 4.7. *For any choice of $\alpha, \gamma \in (0,1)$ we have that $\hat{\beta}_c > 0$ $\mathbb{P}$-a.s., where α is the disorder exponent in Assumption 4.1, while γ is the renewal exponent of Assumption 4.2.*

By using the same technique we complete the result [7, Prop. 2.5] about the structure of β_c, the random threshold defined for the directed polymer model in a random environment with heavy tails (we recall its definition in Section 4.6). Precisely

Theorem 4.8. *Let β_c as in* (4.104)*, then, if $\mathbb{P}_\infty$ denotes the law of the continuum environment,*

(1) *For any $\alpha \in (0, \frac{1}{2})$, $\beta_c > 0$, $\mathbb{P}_\infty$-a.s.*

(2) *For any $\alpha \in [\frac{1}{2}, 2)$, $\beta_c = 0$, $\mathbb{P}_\infty$-a.s.*

Remark 4.9. In [7] the value of β_c was unknown for $\alpha \in (1/3, 1/2)$.

4.1.3 Organization of the Chapter

In rest of the chapter we prove the results of this section. Section 4.2 contains some preliminary definitions and tools that we use for our proofs. Sections 4.3 contains the proof of Theorem 4.5 and Section 4.4 the proof of Theorems 4.4 and 4.6. In Section 4.5 we prove Theorem 4.7 and then in Section 4.6 we recall the definition of the Directed Polymer Model, proving Theorem 4.8.

4.2 Energy & entropy

In this section we define the random sets $I_{\beta,N}$, $\hat{I}_{\hat{\beta},\infty}$ and we motivate the choice of β_N in (4.5).

To define the random set $I_{\beta,N}$ we compare the *Energy* and the *entropy* of a given configuration: for a finite set $I = \{x_0 = 0 < x_1 < \cdots < x_\ell = 1\}$ we define its *Energy* as

$$\sigma_N(I) = \sum_{n=1}^{N-1} \omega_n \mathbb{1}(n/N \in I) \tag{4.11}$$

and its *entropy* as

$$E(I) = \sum_{k=1}^{\ell} (x_i - x_{i-1})^\gamma. \tag{4.12}$$

By using these two ingredients we define

$$I_{\beta,N} = \operatorname*{arg\,max}_{I \in \mathbf{X}^{(N)}} \left(\beta\sigma_N(I) - \mathsf{c}N^\gamma E(I)\right), \tag{4.13}$$

where γ and c are defined in (2) of Assumption 4.2 and $\mathbf{X}^{(N)}$ is the space of all possible subsets of $\{0, 1/N, \cdots, 1\}$ containing 0 and 1.

By using (4.13) we can find the right rescaling for β: indeed it has to be chosen in such a way to make the Energy and the entropy comparable. For this purpose it is convenient to work with a rescaled version of the disorder. We consider $(\tilde{M}_i^{(N)})_{i=1}^{N-1}$ the ordered statistics of $(\omega_i)_{i=1}^{N-1}$ — which means that $\tilde{M}_1^{(N)}$ is the biggest value among $\omega_1, \cdots, \omega_{N-1}$, $\tilde{M}_2^{(N)}$ is the second biggest one and so on — and $(Y_i^{(N)})_{i=1}^{N-1}$ a random permutation of $\{\frac{1}{N}, \cdots 1 - \frac{1}{N}\}$, independent of the ordered statistics. The sequence $((\tilde{M}_i^{(N)}, Y_i^{(N)})_{i=1}^{N-1}$ recovers the disorder $(\omega_i)_{i=1}^{N-1}$. The asymptotic behavior of such sequence is known and it allows us to get the right rescaling of β. Let us recall the main result that we need.

4.2.1 The Disorder

Let us start to note that for any fixed k as $N \to \infty$

$$(Y_i^{(N)})_{i=1,\cdots,k} \xrightarrow{(d)} (Y_i^{(\infty)})_{n=1,\cdots,k}, \tag{4.14}$$

where $(Y_i^{(\infty)})_{i\in\mathbb{N}}$ is an i.i.d. sequence of Uniform($[0,1]$).

For the ordered statistics, from classical extreme value theory, see e.g. [72, Section 1.1], we have that there exists a sequence $(b_N)_N$ such that for any fixed $k > 0$, as $N \to \infty$

$$(M_i^{(N)} := b_N^{-1}\tilde{M}_i^{(N)})_{i=1,\cdots,k} \xrightarrow{(d)} (M_i^{(\infty)})_{n=1,\cdots,k}, \tag{4.15}$$

where $M_i^{(\infty)} = T_i^{-1/\alpha}$, with T_i a sum of i independent exponentials of mean 1 and α is the exponent of the disorder introduced in (4.3). The sequence b_N is characterized by the following relation

$$\mathbb{P}(\omega_1 > b_N) \sim \frac{1}{N}, \quad N \to \infty. \tag{4.16}$$

This implies that $b_N \sim N^{\frac{1}{\alpha}}\ell_0(N)$, where $\ell_0(\cdot)$ is a suitable slowly varying function uniquely defined by $L_0(\cdot)$, cf. (4.3).

We can get a stronger result without a big effort, which will be very useful in the sequel. Let us consider the (independent) sequences $(M_i^{(N)})_{i=1}^{N-1}$ and $(Y_i^{(N)})_{i=1}^{N-1}$ and

$$w_i^{(N)} := \begin{cases} (M_i^{(N)}, Y_i^{(N)})_{i=1}^{N-1}, & i < N, \\ 0, & i \geq N, \end{cases} \tag{4.17}$$

$$w_i^{(\infty)} := (M_i^{(\infty)}, Y_i^{(\infty)})_{i\in\mathbb{N}}, \tag{4.18}$$

We can look at $w^{(N)} = (w_i^{(N)})_{i\in\mathbb{N}}$ and $w^{(\infty)} = (w_i^{(\infty)})_{i\in\mathbb{N}}$ as random variables taking values in $\mathcal{S} := (\mathbb{R}^2)^{\mathbb{N}}$. Let us equip $\mathcal{S}$ with the product topology: a sequence $x^{(N)}$ converges to $x^{(\infty)}$ if and only if for any fixed $i \in \mathbb{N}$ one has $\lim_{N\to\infty} x_i^{(N)} = x_i^{(\infty)}$. In such a way $\mathcal{S}$ is a completely metrizable space and a $\mathcal{S}$-valued random sequence $(w^{(N)})_N$ converges in law to $w^{(\infty)}$ if and only if for any fixed k, the truncated sequence $(w_1^{(N)}, \cdots, w_k^{(N)}, 0, \cdots)$ converges in law to $(w_1^{(\infty)}, \cdots, w_k^{(\infty)}, 0, \cdots)$. Therefore (4.14) and (4.15) imply that

$$w^{(N)} \xrightarrow{(d)} w^{(\infty)}, \quad N \to \infty \tag{4.19}$$

in $\mathcal{S}$. Henceforth we refer to $w^{(N)}$ as the Discrete Disorder of size N, and to $w^{(\infty)}$ as the Continuum Disorder.

4.2.2 The Energy

Recalling (4.11) we define the rescaled discrete Energy function $\hat{\sigma}_N : \mathbf{X} \to \mathbb{R}_+$ as

$$\hat{\sigma}_N(\cdot) = \frac{\sigma_N(\cdot)}{b_N} = \sum_{i=1}^{N-1} M_i^{(N)} \mathbb{1}(Y_i^{(N)} \in \cdot), \tag{4.20}$$

and (4.13) becomes

$$I_{\frac{N^\gamma}{b_N}\beta,N} = \arg\max_{I\in\mathbf{X}^{(N)}} \left(\beta\hat{\sigma}_N(\cdot) - \mathsf{c}E(I)\right), \tag{4.21}$$

Therefore we choose β_N such that

$$\hat{\beta}_N := \frac{b_N}{N^\gamma}\beta_N \tag{4.22}$$

converges to $\hat{\beta} \in (0,\infty)$. This is equivalent to relation (4.5). Since in the sequel we will study the set $I_{\frac{N^\gamma}{b_N}\beta,N}$, it is convenient to introduce the notation

$$\hat{I}_{\beta,N} = I_{\frac{N^\gamma}{b_N}\beta,N}. \tag{4.23}$$

In particular $\hat{I}_{\hat{\beta}_N,N} = I_{\beta_N,N}$.

Remark 4.10. Let us stress that the value of c is inessential and it can be included in the parameter $\hat{\beta}$ by a simple rescaling. Therefore from now on we assume c = 1.

It is essential for the sequel to extend the definition of $\hat{I}_{\beta,N}$ to the whole space $\mathbf{X}$ equipped with the Hausdorff metric. This generalization leads us to define the same kind of random set introduced in (4.21) in which we use suitable continuum Energy and entropy.

We define the continuum Energy Function $\hat{\sigma}_\infty : \mathbf{X} \to \mathbb{R}_+$ as

$$\hat{\sigma}_\infty(\cdot) = \sum_{i=1}^{\infty} M_i^{(\infty)} \mathbb{1}(Y_i^{(\infty)} \in \cdot), \tag{4.24}$$

where $(M_i^{(\infty)})_{i\in\mathbb{N}}$ and $(Y_i^{(\infty)})_{i\in\mathbb{N}}$ are the two independent random sequences introduced in (4.14) and (4.15).

Remark 4.11. Let us observe that $\hat{\sigma}_\infty(I) < \infty$ for all $I \in \mathbf{X}$, because the serie $\sum_{i=1}^{\infty} M_i^{(\infty)}$ converges a.s. Indeed, the law of large numbers ensures that a.s. $M_i^{(\infty)} \sim i^{-\frac{1}{\alpha}}$ as $i \to \infty$, cf. its definition below (4.15), and $\alpha \in (0,1)$.

We conclude this section by proving that $\hat{\sigma}_N$, with $N \in \mathbb{N} \cup \{\infty\}$, is an upper semi-continuous function. For this purpose, for $k, N \in \mathbb{N} \cup \{\infty\}$ we define the k-truncated Energy function as

$$\hat{\sigma}_N^{(k)}(\cdot) = \sum_{i=1}^{(N-1)\wedge k} M_i^{(N)} \mathbb{1}(Y_i^{(N)} \in \cdot). \tag{4.25}$$

Let us stress that the support of $\hat{\sigma}_N^{(k)}$ is given by the space of all possible subsets of $Y^{(N,k)}$, the set of the first k-maxima positions

$$Y^{(N,k)} = \{Y_i^{(N)}, i = 1,2,3,\cdots,(N-1)\wedge k\} \cup \{0,1\}. \tag{4.26}$$

Whenever $k \geq N$ we write simply $Y^{(N)}$.

Theorem 4.12. *For any fixed $k, N \in \mathbb{N} \cup \{\infty\}$ and for a.e. realization of the disorder $w^{(N)}$, the function $\hat{\sigma}_N^{(k)} : \mathbf{X} \to \mathbb{R}_+$ is upper semi-continuous (u.s.c.).*

Remark 4.13. For sake of clarity let us underline that in the Hausdorff metric, cf. (4.6), $d_H(A,B) < \varepsilon$ if and only if for any $x_1 \in A$ there exists $x_2 \in B$ such that $|x_1 - x_2| < \varepsilon$ and vice-versa switching the role of A and B.

Proof. Let us start to consider the case $N \wedge k < \infty$. For a given $I_0 \in \mathbf{X}$, let ι be the set of all points of $Y^{(N,k)}$ which are not in I_0. Since $Y^{(N,k)}$ has a finite number of points there exists $\eta > 0$ such that $d(z, I_0) > \eta$ for any $z \in \iota$. Then if $I \in \mathbf{X}$ is sufficiently close to I_0, namely $d_H(I, I_0) \leq \eta/2$, then $d(z,I) > \eta/2 > 0$ for any $z \in \iota$. Therefore, among the first k-maxima, I can at most hit only the points hit by I_0, namely $\hat{\sigma}_N^{(k)}(I) \leq \hat{\sigma}_N^{(k)}(I_0)$ and this concludes the proof of this first part.

For the case $N \wedge k = \infty$ it is enough to observe that the difference between the truncated Energy and the original one

$$\sup_{I\in\mathbf{X}} \left|\hat{\sigma}_\infty(I) - \hat{\sigma}_\infty^{(k)}(I)\right| = \sup_{I\in\mathbf{X}} \left|\sum_{i=1}^{\infty} M_i^{(\infty)} \mathbb{1}(Y_i^{(\infty)} \in I) - \sum_{i=1}^{k} M_i^{(\infty)} \mathbb{1}(Y_i^{(\infty)} \in I)\right| \leq \sum_{i>k} M_i^{(\infty)}, \tag{4.27}$$

converges to 0 as $k \to \infty$ because $\sum_i M_i^{(\infty)}$ is a.s. finite, cf. Remark 4.11. Therefore the sequence of u.s.c. functions $\hat{\sigma}_\infty^{(k)}$ converges uniformly to $\hat{\sigma}_\infty$ and this implies the u.s.c. of the limit. □

4.2.3 The entropy

Let us define

$$\mathbf{X}^{(\mathrm{fin})} = \{I \in \mathrm{X} : |I| < \infty\} \tag{4.28}$$

and remark that it is a countable dense subset of $\mathbf{X}$ with respect to the Hausdorff Metric.

For a given set $I = \{x_0 < x_1 < \cdots < x_\ell\} \in \mathbf{X}^{(\mathrm{fin})}$ we define the *entropy* as

$$E(I) = \sum_{k=1}^{\ell} (x_i - x_{i-1})^\gamma. \tag{4.29}$$

Theorem 4.14. *The following hold*

(1) *The entropy $E(\cdot)$ is strictly increasing with respect to the inclusion of finite sets, namely if $I_1, I_2 \in \mathbf{X}^{(fin)}$ and $I_1 \subsetneq I_2$, then $E(I_2) > E(I_1)$,*

(2) *The function $E : \mathbf{X}^{(fin)} \to \mathbb{R}_+$ is lower semi continuous (l.s.c.).*

Proof. To prove (1) let us note that if $I_2 = \{0, a_1, x, a_2, 1\}$ and $I_2 = \{0, a_1, a_2, 1\}$, with $0 \le a_1 < x < a_2 \le 1$ then $E(I_2) - E(I_1) = (x - a_1)^\gamma + (a_2 - x)^\gamma - (a_2 - a_1)^\gamma > 0$ because $\gamma < 1$, thus $a^\gamma + b^\gamma > (a + b)^\gamma$ for any $a, b > 0$. The claim for the general case follows by a simple induction argument.

To prove (2) we fix $I_0 \in \mathbf{X}^{(\mathrm{fin})}$ and we show that if $(I_n)_n$ is a sequence of finite set converging (in the Hausdorff metric) to I_0, then it must be $\liminf_{n\to\infty} E(I_n) \ge E(I_0)$ and by the arbitrariness of the sequence the proof will follow.

Let $I_0 \in \mathbf{X}^{(\mathrm{fin})}$ be fixed and let us observe that if we fix $\varepsilon > 0$ small (precisely smaller than the half of the minimum of the distance between the points of I_0), then by Remark 4.13 any set I for which $d_H(I, I_0) < \varepsilon$ must have at least the same number of points of I_0, i.e. $|I| \ge |I_0|$. In such a way if (I_n) is a sequence of finite sets converging to I_0, then for any n large enough we can pick out a subset I'_n of I_n with the same number of points of I_0 such that $(I'_n)_n$ converges to I_0. Necessary the points of I'_n converge to the ones of I_0, so that $\lim_{n\to\infty} E(I'_n) = E(I_0)$. By using Part (1) we have that for any n, $E(I_n) \ge E(I'_n)$, so that $\liminf_{n\to\infty} E(I_n) \ge E(I_0)$ and the proof follows. □

We are now ready to define the entropy of a generic set $I \in \mathbf{X}$. The goal is to obtain an extension which conserves the properties of the entropy E on $\mathbf{X}^{(\mathrm{fin})}$, cf. Theorem 4.14. This extension is not trivial because E is strictly l.s.c., namely given $I \in \mathbf{X}^{(\mathrm{fin})}$ it is always possible to find two sequences $(I_N^{(1)})_N, (I_N^{(2)})_N \in \mathbf{X}^{(\mathrm{fin})}$ converging to I such that $\lim_{N\to\infty} E(I_N^{(1)}) = E(I)$ and $\lim_{N\to\infty} E(I_N^{(2)}) = \infty$. For instance let us consider the simplest case, when $I = \{0, 1\}$. Then we may consider $I_N^{(1)} \equiv I$ for any N, so that $E(I_N^{(1)}) \equiv E(\{0, 1\})$, and $I_N^{(2)}$ the set made by $2N$ points such that the first N are equispaced in a neighborhood of 0 of radius $N^{-\varepsilon}$ and the others N in a neighborhood of 1 always of radius $N^{-\varepsilon}$, with $\varepsilon = \varepsilon(\gamma)$ small. Then $I_N^{(2)} \to I$ as $N \to \infty$ and $E(I_N^{(2)}) = 2N \cdot 1/N^{\gamma(1+\varepsilon)} + (1 - 2/N^\varepsilon)^\gamma = O(N^{1-\gamma(1+\varepsilon)}) \to \infty$ as $N \to \infty$ if $\varepsilon < (1 - \gamma)/\gamma$.

In order to avoid this problem for $I \in \mathbf{X}$ we define

$$\bar{E}(I) = \liminf_{J \to I, J \in \mathbf{X}^{(\mathrm{fin})}} E(J). \tag{4.30}$$

Let us stress that $\bar{E}$ is nothing but the smallest l.s.c. extension of E to the whole space $\mathbf{X}$, see e.g. [17, Prop. 5 TG IV.31].

Theorem 4.15. *The following hold:*

(1) *The function $\bar{E}(\cdot)$ is increasing with respect to the inclusion of sets, namely if $I_1, I_2 \in \mathbf{X}$ with $I_1 \subset I_2$ then $\bar{E}(I_2) \geq \bar{E}(I_1)$.*

(2) *The function $\bar{E} : \mathbf{X} \to \mathbb{R}_+$ is l.s.c. and $\bar{E}\,|_{\mathbf{X}^{(fin)}} \equiv E$.*

Remark 4.16. To be more clear we recall that

$$\bar{E}(I) = \liminf_{J \to I, J \in \mathbf{X}^{(\text{fin})}} E(J) := \sup_{\delta > 0} \left[\inf \left\{ E(J) : J \in B_H(I, \delta) \cap \mathbf{X}^{(\text{fin})} \backslash \{I\} \right\} \right], \tag{4.31}$$

where $B_H(I, \delta)$ denotes the disc of radius δ centered on I in the Hausdorff Metric.

If $\bar{E}(I) \in \mathbb{R}$ such definition is equivalent to say

(a) For any $\varepsilon > 0$ and for any $\delta > 0$ there exists $J \in B_H(\delta, I) \cap \mathbf{X}^{(\text{fin})} \backslash \{I\}$ such that $\bar{E}(I) + \varepsilon > E(J)$.

(b) For any $\varepsilon > 0$ there exists $\delta_0 > 0$ such that for any $J \in B_H(\delta_0, I) \cap \mathbf{X}^{(\text{fin})} \backslash \{I\}$, $E(J) > \bar{E}(I) - \varepsilon$.

Note that (a) expresses the property to be an infimum, while (b) corresponds to be a supremum.

Proof of Theorem 4.15. We have only to prove (1). Let $I, J \in \mathbf{X}$ such that $J \subset I$. If $\bar{E}(I) = \infty$ there is nothing to prove, therefore we can assume that $\bar{E}(I) \in \mathbb{R}$.

Let us fix $\varepsilon > 0$ and $\delta > 0$ (which will be chosen in the sequel). By (a) there exists $I' \in \mathbf{X}^{(\text{fin})}$ such that $\bar{E}(I) + \varepsilon \geq E(I')$ and $d_H(I, I') < \delta$. By the definition of the Hausdorff metric, the family of discs of radius δ indexed by I' — $(B(x, \delta))_{x \in I'}$ — covers I and thus also J. Therefore if $J' \subset I'$ is the minimal cover of J obtained from I', i.e. $J' := \min\{L \subset I' : J \subset \cup_{x \in L} B(x, \delta)\}$, then it must hold that $d_H(J, J') < \delta$. By Theorem 4.14 it follows that $E(I') \geq E(J')$ and thus $\bar{E}(I) + \varepsilon \geq E(J')$. Let us consider $\bar{E}(J)$ and take $\delta_0 > 0$ as prescript in (b), then as soon as $\delta < \delta_0$, it must hold that $E(J') \geq \bar{E}(J) - \varepsilon$ and this concludes the proof. □

From now on in order to simplify the notation we use E instead of $\bar{E}$ to indicate the function E defined on all $\mathbf{X}$.

Corollary 4.17. *Let $I \in \mathbf{X}$ such that $E(I) < \infty$. Let $x \notin I$, then $E(I \cup \{x\}) > E(I)$. It follows that the function E is strictly increasing whenever it is finite: if $I \subsetneq J$ and $E(I) < \infty$, then $E(I) < E(J)$.*

Proof. Let $I \in \mathbf{X}$ and let us assume that $E(I) < \infty$. Note that $x \notin I$ means that there exists $\delta > 0$ such that $I \cap (x - \delta, x + \delta) = \emptyset$ because I is closed. We consider a, b the left and right closest points to x in I. Then the proof will follow by proving that

$$E(I \cup \{x\}) - E(I) \geq (x - a)^\gamma + (b - x)^\gamma - (b - a)^\gamma, \tag{4.32}$$

because the r.h.s. is a quantity strictly bigger than 0, since $\gamma < 1$.

To prove (4.32), we show that the result is true for any finite set in an ε-neighborhood (in the Hausdorff metric) of $I \cup \{x\}$ and then we deduce the result for $E(I)$, by using its definition (4.30). Let us start to observe that for any ε small enough, if A is a set in an ε-neighborhood of $I \cup \{x\}$, then it can be written as union of two disjoint sets D, C where D is in a ε-neighborhood of I and C in a ε-neighborhood of $\{x\}$. In particular this holds when A is a finite set, and thus

$$B_H(I \cup \{x\}, \varepsilon) \cap \mathbf{X}^{(\text{fin})} = \{A \in \mathbf{X}^{(\text{fin})} \ : \ A = D \cup C,\ D \in B_H(I, \varepsilon) \text{ and } C \in B_H(\{x\}, \varepsilon)\}.$$

Furthermore, we can partition any such D in two disjoint sets $D' = D \cap [0, x)$ and $D'' = (x, 1]$.

For a fixed set $S \in \mathbf{X}$, let l_S be its smallest point bigger than 0 and r_S its biggest point smaller than 1. By using this notation it follows from the definition of the entropy of a finite set (4.29) that for any such $A \in B_H(I \cup \{x\}, \varepsilon) \cap \mathbf{X}^{(\text{fin})}$ we have

$$E(A) = E(D \cup C) = E(D) - (l_{D''} - r_{D'})^\gamma + E(C \cup \{0, 1\}) - l_C^\gamma - (1 - r_C)^\gamma + (l_C - r_{D'})^\gamma + (l_{D''} - r_C)^\gamma. \tag{4.33}$$

By Theorem 4.14 we can bound $E(C \cup \{0,1\}) \geq l_C^{\gamma} + (1 - r_C)^{\gamma} + (r_C - l_C)^{\gamma}$. Putting such expression in (4.32) we obtain

$$\begin{aligned} E(A) = E(D \cup C) \geq \\ E(D) - (l_{D''} - r_{D'})^{\gamma} + (l_C - r_{D'})^{\gamma} + (l_{D''} - r_C)^{\gamma} + (r_C - l_C)^{\gamma} \geq E(D) + e(\varepsilon), \end{aligned} \tag{4.34}$$

where

$$e(\varepsilon) = \inf\{(l_C - r_{D'})^{\gamma} + (l_{D''} - r_C)^{\gamma} + (r_C - l_C)^{\gamma} - (l_{D''} - r_{D'})^{\gamma}\}.$$

Such inf is taken among all possible $D = D' \cup D'' \in B_H(I, \varepsilon) \cap \mathbf{X}^{(\mathtt{fin})}$ and $C \in B_H(\{x\}, \varepsilon)\} \cap \mathbf{X}^{(\mathtt{fin})}$.

Finally (4.34) implies that $\inf E(A) \geq \inf E(D) + e(\varepsilon)$, where the inf is taken among all possible $A = D \cup C \in B_H(I \cup \{x\}, \varepsilon) \cap \mathbf{X}^{(\mathtt{fin})} \backslash \{I \cup \{x\}\}$. By taking the limit for $\varepsilon \to 0$ we have $e(\varepsilon) \to (x-a)^{\gamma} + (b-x)^{\gamma} - (b-a)^{\gamma}$ and the result follows by (4.31), since the r.h.s. of (4.34) is independent of C. □

Proposition 4.18. *For any* $0 \leq a < b \leq 1$ *we have that* $E([a,b]) = \infty$.

Proof. Let us consider the case in which $a = 0, b = 1$, the other cases follow in a similar way. By Theorem 4.14 we have that $E([0,1]) \geq E(\{0, 1/N, \cdots, 1\}) = N^{1-\gamma} \uparrow \infty$ as $N \uparrow \infty$ because $\gamma < 1$. □

4.2.4 The Energy-entropy

Definition 4.19. For any $N, k \in \mathbb{N} \cup \{\infty\}$ and $\beta \in (0, \infty)$ we define, cf. (4.25) and (4.30),

$$U_{\beta,N}^{(k)}(I) = \beta \hat{\sigma}_N^{(k)}(I) - E(I). \tag{4.35}$$

Note that $U_{\beta,N}^{(k)}$ is upper semi-continuous on $(\mathbf{X}, d_H)$, a compact metric space, therefore its maximizer

$$\hat{u}_{\beta,N}^{(k)} = \max_{I \in \mathbf{X}} U_{\beta,N}^{(k)}(I). \tag{4.36}$$

is well defined.

Whenever $k \geq N$ we will omit the superscript (k) from the notation.

Theorem 4.20. *For any* $N, k \in \mathbb{N} \cup \{\infty\}$, $\beta > 0$ *and for a.e. realization of the disorder* $w^{(N)}$, *the maximum* $\hat{u}_{\beta,N}^{(k)}$ *is achieved in only one set, i.e. the solution at*

$$\hat{I}_{\beta,N}^{(k)} = \arg\max_{I \in \mathbf{X}} U_{\beta,N}^{(k)}(I) \tag{4.37}$$

is unique. Moreover for any $N \in \mathbb{N}$ *we have that* $\hat{I}_{\beta,N}^{(k)} \in \mathbf{X}^{(N)}$.

Proof. We claim that if I is a solution of (4.37), then by using Corollary 4.17

$$I \subset Y^{(N,k)}, \quad \text{if } N \wedge k < \infty, \tag{4.38}$$

$$I = \overline{I \cap Y^{(\infty)}} \quad \text{if } N \wedge k = \infty. \tag{4.39}$$

Indeed if $N \wedge k < \infty$ and (4.38) fails, then there exists $x \in I$ such that $x \notin Y^{(N,k)}$ and this implies $\hat{\sigma}_N^{(k)}(I) = \hat{\sigma}_N^{(k)}(I - \{x\})$, but $E(I - \{x\}) < E(I)$ by Corollary 4.17. Therefore $U_{\beta,N}^{(k)}(I - \{x\}) > U_{\beta,N}^{(k)}(I) = \hat{u}_{\beta,N}^{(k)}$, a contradiction. The case $N \wedge k = \infty$ follows in an analogous way always by using Corollary 4.17, because the set in the r.h.s. of (4.39), which is a subset of I, has the same Energy as I but smaller entropy. Now we are able to conclude the uniqueness, by following the same ideas used in [50, Proposition 4.1] or [7, Lemma 4.1]: let I^1, I^2 be two subsets achieving the maximum. By using

(4.38) and (4.39) if $I^1 \neq I^2$, then there would exist $Y_j^{(N)}$ such that $Y_j^{(N)} \in I_1$ and $Y_j^{(N)} \notin I_2$. Note that if $N \wedge k = \infty$, by (4.39) we can assume $Y_j^{(N)} \in Y^{(\infty)}$, so that

$$\max_{I:Y_j^{(N)} \in I} U_{\beta,N}^{(k)}(I) = \max_{I:Y_j^{(N)} \notin I} U_{\beta,N}^{(k)}(I) \tag{4.40}$$

and this leads to

$$\beta M_j^{(N)} = \beta \hat{\sigma}_N^{(k)}(Y_j^{(N)}) = \max_{I:Y_j^{(N)} \notin I} U_{\beta,N}^{(k)}(I) - \max_{I:Y_j^{(N)} \in I} \left\{ \beta \sum_{k \neq j: Y_k^{(N)} \in I} M_k^{(N)} - E(I) \right\}. \tag{4.41}$$

Let us stress that the r.h.s. is independent of $M_j^{(N)}$, which is in the l.h.s. Then, by conditioning on the values of $(M_i^{(N)})_{i \in \mathbb{N}, i \neq j}$ and $(Y_i^{(N)})_{i \in \mathbb{N}}$ we have that the l.h.s. has a continuous distribution, while the r.h.s. is a constant, so that the event in which the r.h.s. is equal to the l.h.s. has zero probability. By countable sub-additivity of the probability we have that a.s. $I_1 = I_2$. □

4.3 Convergence

The aim of this section is to discuss the convergence of $\hat{I}_{\hat{\beta}_N,N}$, (4.37), and $\hat{u}_{\hat{\beta}_N,N}$, (4.36), when $\lim_{N\to\infty} \hat{\beta}_N = \hat{\beta} \in (0, \infty)$, cf. (4.22).

For technical convenience we build a coupling between the discrete disorder and the continuum one. We recall that by (4.19) $w^{(N)}$ converges in distribution to $w^{(\infty)}$ on $\mathcal{S}$, a completely metrizable space. Therefore by using Skorokhod's representation Theorem (see [13, theorem 6.7]) we can define $w^{(N)}$ and $w^{(\infty)}$ on a common probability space in order to assume that their convergence holds almost surely.

Lemma 4.21. *There is a coupling (that, with a slight abuse of notation, we still call $\mathbb{P}$) of the continuum model and the discrete one, under which*

$$w^{(N)} = (M_i^{(N)}, Y_i^{(N)})_{i \in \mathbb{N}} \xrightarrow[\mathbb{P}-a.s.]{\mathcal{S}} w^{(\infty)} = (M_i^{(\infty)}, Y_i^{(\infty)})_{i \in \mathbb{N}}, \text{ as } N \to \infty. \tag{4.42}$$

In particular for any fixed $\varepsilon, \delta > 0$ and $k \in \mathbb{N}$ there exists $\hat{N} < \infty$ such that for all $N > \hat{N}$

$$\mathbb{P}\left(\sum_{j=1}^{(N-1)\wedge k} | M_j^{(N)} - M_j^{(\infty)} | < \varepsilon \right) > 1 - \delta, \tag{4.43}$$

$$\mathbb{P}\left(\sum_{j=1}^{(N-1)\wedge k} | Y_j^{(N)} - Y_j^{(\infty)} | < \varepsilon \right) > 1 - \delta, \tag{4.44}$$

4.3.1 Convergence Results

Let us rewrite an equivalent, but more handy definition of $\hat{I}_{\beta,N}^{(k)}$ and $\hat{u}_{\beta,N}^{(k)}$: for a given $k \in \mathbb{N}$ let

$$C_k = \{A : A \subset \{1, \cdots, k\}\} \tag{4.45}$$

and for any $k \in \mathbb{N}$, $N \in \mathbb{N} \cup \{\infty\}$ and $A \subset \{1, \cdots, k\}$ let $Y_A^{(N)} = \{Y_i^{(N)}\}_{i \in A} \cup \{0, 1\}$, which is well defined also for $A = \emptyset$. Therefore by Theorem 4.20 we can write

$$\begin{aligned} \hat{u}_{\beta,N}^{(k)} &= \max_{A \in C_k} \left[\beta \sum_{i \in A} M_i^{(N)} - E(Y_A) \right], \\ \hat{I}_{\beta,N}^{(k)} &= Y^{(N)}_{A^{(k)}_{\beta,N}}, \end{aligned} \tag{4.46}$$

for a suitable random set of indexes $A_{\beta,N}^{(k)}$ (which can be empty or not). We have our first convergence result.

Proposition 4.22. *Assume that $\hat{\beta}_N \to \hat{\beta}$ as $N \to \infty$. Then for any fixed $\delta > 0$ and $k \in \mathbb{N}$ there exists N_k such that for any $N > N_k$*

$$\mathbb{P}\left(A_{\hat{\beta}_N,N}^{(k)} = A_{\hat{\beta},\infty}^{(k)} \right) > 1 - \delta. \tag{4.47}$$

Proof. To prove the claim by using the sub-additivity of the probability, it is enough to prove that for any $r \in \{1, \cdots, k\}$

$$\mathbb{P}\left(r \notin A_{\hat{\beta}_N,N}^{(k)}, r \in A_{\hat{\beta},\infty}^{(k)} \right) \to 0, \quad N \to \infty, \tag{4.48}$$

$$\mathbb{P}\left(r \in A_{\hat{\beta}_N,N}^{(k)}, r \notin A_{\hat{\beta},\infty}^{(k)} \right) \to 0, \quad N \to \infty. \tag{4.49}$$

We detail the first one, the second one follows in an analogous way. On the event $\{r \notin A_{\hat{\beta}_N,N}^{(k)}, r \in A_{\hat{\beta},\infty}^{(k)}\}$ we consider

$$\hat{u}_{(r)} := \max_{A \in C_k, r \notin A} \left[\hat{\beta} \sum_{i \in A} M_i^{(\infty)} - E(Y_A) \right] < \hat{u}_{\hat{\beta},\infty}^{(k)} \tag{4.50}$$

because $r \in A_{\hat{\beta},\infty}^{(k)}$ and the set that achieves the maximum is unique. Then

$$\begin{aligned} &\max_{A \in C_k, r \notin A} \left[\hat{\beta}_N \sum_{i \in A} M_i^{(N)} - E(Y_A) \right] \\ &\qquad \leq \max_{A \in C_k, r \notin A} \left[\hat{\beta} \sum_{i \in A} M_i^{(\infty)} - E(Y_A) \right] + |\hat{\beta}_N - \hat{\beta}| \sum_{i=1}^k M_i^{(N)} + \hat{\beta} \sum_{i=1}^k |M_i^{(N)} - M_i^{(\infty)}| \\ &\qquad = \hat{u}_{(r)} + |\hat{\beta}_N - \hat{\beta}| \sum_{i=1}^k M_i^{(N)} + \hat{\beta} \sum_{i=1}^k |M_i^{(N)} - M_i^{(\infty)}| \end{aligned} \tag{4.51}$$

and in the same way, always on the event $\{r \notin A_{\hat{\beta}_N,N}^{(k)}, r \in A_{\hat{\beta},\infty}^{(k)}\}$,

$$\max_{A \in C_k, r \in A} \left[\hat{\beta}_N \sum_{i \in A} M_i^{(N)} - E(Y_A) \right] \geq \hat{u}_{\hat{\beta},\infty}^{(k)} - |\hat{\beta}_N - \hat{\beta}| \sum_{i=1}^k M_i^{(N)} - \hat{\beta} \sum_{i=1}^k |M_i^{(N)} - M_i^{(\infty)}|. \tag{4.52}$$

Therefore by using the assumption that $r \notin A_{\hat{\beta}_N,N}^{(k)}$ we have that the l.h.s. of (4.51) is larger than the l.h.s. of (4.52). Together with (4.50) we obtain $0 < \hat{u}_{\hat{\beta},\infty}^{(k)} - \hat{u}_{(r)} \leq 2\hat{\beta} \sum_{i=1}^k |M_i^{(N)} - M_i^{(\infty)}| + 2|\hat{\beta}_N - \hat{\beta}| \sum_{i=1}^k M_i^{(N)}$, and a simple inclusion of events gives

$$\mathbb{P}\left(r \notin A_{\hat{\beta}_N,N}^{(k)}, r \in A_{\hat{\beta},\infty}^{(k)} \right) \leq \mathbb{P}\left(0 < \hat{u}_{\hat{\beta},\infty}^{(k)} - \hat{u}_{(r)} \leq 2\hat{\beta} \sum_{i=1}^k |M_i^{(N)} - M_i^{(\infty)}| + 2|\hat{\beta}_N - \hat{\beta}| \sum_{i=1}^k M_i^{(N)} \right). \tag{4.53}$$

The proof follows by observing that the r.h.s. converges to 0 as $N \to \infty$ by Lemma 4.21. $\square$

The following proposition contains the convergence results for the truncated quantities $\hat{I}^{(k)}_{\hat{\beta}_N,N}$ and $\hat{u}^{(k)}_{\hat{\beta}_N,N}$, cf. (4.37) and (4.36) respectively.

We introduce the maximum of $U^{(k)}_{\beta,N}$, cf. (4.35), outside a neighborhood of radius δ of $\hat{I}^{(k)}_{\beta,N}$

Definition 4.23. For any $\delta > 0, \beta \in (0,\infty)$ we define

$$\hat{u}^{(k)}_{\beta,N}(\delta) = \max_{I \in \mathbf{X}: d_H(I,\hat{I}^{(k)}_{\beta,N}) \geq \delta} U^{(k)}_{\beta,N}(I), \tag{4.54}$$

where $U^{(k)}_{\beta,N}$ is defined in (4.35).

Proposition 4.24. *Assume that $\hat{\beta}_N \to \hat{\beta}$ as $N \to \infty$. The following hold*

(1) *For every fixed $\delta > 0$, $\beta \in (0,\infty)$ $\mathbb{P}\left(\liminf_{k\to\infty}(\hat{u}^{(k)}_{\beta,\infty} - \hat{u}^{(k)}_{\beta,\infty}(\delta)) > 0\right) = 1.$*

(2) *For any $\varepsilon, \delta > 0$ and for any fixed k there exists N_k such that $\mathbb{P}\left(|\hat{u}^{(k)}_{\hat{\beta}_N,N} - \hat{u}^{(k)}_{\hat{\beta},\infty}| < \varepsilon\right) > 1-\delta$, for any $N > N_k$.*

(3) *For any $\varepsilon, \delta > 0$ and for any fixed k there exists N_k such that $\mathbb{P}\left(d_H(\hat{I}^{(k)}_{\hat{\beta}_N,N}, \hat{I}^{(k)}_{\hat{\beta},\infty}) < \varepsilon\right) > 1-\delta$, for any $N > N_k$.*

(4) *For any $\varepsilon, \delta > 0$, there exist $\eta, K > 0$ and $(N_k)_{k>K}$, such that $\mathbb{P}\left(\hat{u}^{(k)}_{\hat{\beta}_N,N}(\varepsilon) < \hat{u}^{(k)}_{\hat{\beta}_N,N} - \eta\right) > 1-\delta$, for any $k > K$, and $N > N_k$.*

Proof. We follow [7, Part (3,4) of Proof of Lemma 4.1]. By contradiction if there exists $\delta > 0$ such that $\liminf_{k\to\infty}(\hat{u}^{(k)}_{\beta,\infty} - \hat{u}^{(k)}_{\beta,\infty}(\delta)) = 0$, then we may find a sequence I_{k_j} such that $\limsup_{j\to\infty} U^{(k_j)}_{\beta,\infty}(I_{k_j}) \geq \liminf_{j\to\infty} U^{(k_j)}_{\beta,\infty}(\hat{I}^{(k_j)}_{\beta,\infty})$ and $d_H(\hat{I}^{k_j}_{\beta,\infty}, I_{k_j}) > \delta$. By compactness of the space $\mathbf{X}$ we can suppose that there exists $I_0 \in \mathbf{X}$ such that $\lim_{j\to\infty} I_{k_j} = I_0$, therefore by using the u.s.c. property of $U^{(k)}_{\beta,N}$, cf. Section 4.2.4, that for any fixed $k \in \mathbb{N}$, $U_{\beta,\infty}(I) \geq U^{(k)}_{\beta,\infty}(I)$ and $U^{(k)}_{\beta,\infty}(I) \uparrow U_{\beta,\infty}(I)$ as $k \uparrow \infty$, we get

$$\begin{aligned} U_{\beta,\infty}(I_0) &\geq \limsup_{j\to\infty} U_{\beta,\infty}(I_{k_j}) \geq \limsup_{j\to\infty} U^{(k_j)}_{\beta,\infty}(I_{k_j}) \geq \\ &\geq \liminf_{j\to\infty} U^{(k_j)}_{\beta,\infty}(\hat{I}^{(k_j)}_{\beta,\infty}) \geq \liminf_{j\to\infty} U^{(k_j)}_{\beta,\infty}(\hat{I}_{\beta,\infty}) = U_{\beta,\infty}(\hat{I}_{\beta,\infty}) = \hat{u}_{\beta,\infty}, \end{aligned} \tag{4.55}$$

namely, $U_{\beta,\infty}(I_0) = \hat{u}_{\beta,\infty}$. The uniqueness of the maximizer, cf. Theorem 4.20, implies $I_0 = \hat{I}_{\beta,\infty}$. Thus if we show that $\lim_{k\to\infty} \hat{I}^{(k)}_{\beta,\infty} = \hat{I}_{\beta,\infty}$, then we obtain the desired contradiction, because the two sequences $(I_{k_j})_j$ and $(\hat{I}^{(k_j)}_{\beta,\infty})_j$ are at distance at least δ therefore they cannot converge to the same limit. By compactness of $\mathbf{X}$ we can assume that $\hat{I}^{(k)}_{\beta,\infty}$ converges to I_1. Therefore, again by u.s.c. of $U_{\beta,\infty}$, we get

$$U_{\beta,\infty}(I_1) \geq \limsup_{k\to\infty} U_{\beta,\infty}(\hat{I}^{(k)}_{\beta,\infty}) \geq \limsup_{k\to\infty} U^{(k)}_{\beta,\infty}(\hat{I}^{(k)}_{\beta,\infty}) \geq \limsup_{k\to\infty} U^{(k)}_{\beta,\infty}(\hat{I}_{\beta,\infty}) = U_{\beta,\infty}(\hat{I}_{\beta,\infty}). \tag{4.56}$$

The uniqueness of the maximizer forces $\hat{I}_{\beta,\infty} = I_1$ and this concludes the proof. $\square$

To prove Part (2) we observe that

$$\hat{u}^{(k)}_{\hat\beta,\infty} = \max_{A\in C_k}\left[\hat\beta \sum_{i\in A} M_i^{(\infty)} - E(Y_A)\right] \le \hat{u}^{(k)}_{\hat\beta_N,N} + |\hat\beta_N - \hat\beta| \sum_{i=1}^k M_i + \hat\beta_N \sum_{i=1}^k |M_i^{(\infty)} - M_i^{(N)}|, \tag{4.57}$$

$$\hat{u}^{(k)}_{\hat\beta_N,N} = \max_{A\in C_k}\left[\hat\beta_N \sum_{i\in A} M_i^{(N)} - E(Y_A)\right] \le \hat{u}^{(k)}_{\hat\beta,\infty} + |\hat\beta_N - \hat\beta| \sum_{i=1}^k M_i^{(N)} + \hat\beta \sum_{i=1}^k |M_i^{(\infty)} - M_i^{(N)}|. \tag{4.58}$$

and the proof follows by Lemma 4.21 and the assumption on $\hat\beta_N$. □

To prove Part (3) we observe that by Lemma 4.21 for any fixed $\varepsilon, \delta > 0$ and $k \in \mathbb{N}$, there exists N_k such that, for all $N > N_k$, $\mathbb{P}\left(d(Y_i^{(k)}, Y_i^{(\infty)}) < \varepsilon, \text{ for any } i = 1, \cdots, k\right) > 1 - \delta/2$. By Proposition 4.22 we can furthermore suppose that for any $N > N_k$, $\mathbb{P}\left(A^{(k)}_{\hat\beta_N,N} = A^{(k)}_{\hat\beta,\infty}\right) > 1 - \delta/2$, cf. (4.46). The intersection of such events gives the result. □

To prove Part (4) we prove first an intermediate result: for any given $\delta, \varepsilon, \eta > 0$, and $k \in \mathbb{N}$ there exists N_k such that

$$\hat{u}^{(k)}_{\hat\beta_N,N}(\varepsilon) < \hat{u}^{(k)}_{\hat\beta,\infty}(\varepsilon/4) + \eta/4, \tag{4.59}$$

with probability larger than $1 - \delta/2$, for all $N > N_k$.

For this purpose, by Part (3), for any $k > 0$ there exists $N_k > 0$ such that for all $N > N_k$, $d_H(\hat{I}^{(k)}_{\hat\beta_N,N}, \hat{I}^{(k)}_{\hat\beta,\infty}) < \frac{\varepsilon}{4}$ with probability larger than $1 - \delta/4$. Let I be a set achieving $\hat{u}^{(k)}_{\hat\beta_N,N}(\varepsilon)$, so that by definition $d_H(I, \hat{I}^{(k)}_{\hat\beta_N,N}) \ge \varepsilon$. It is not difficult to see that $I \subset Y^{(N,k)}$ (points outside $Y^{(N,k)}$ does not contribute to the Energy, but increase the entropy). We claim that for any $\eta > 0$ there exists $I' \subset Y^{(\infty,k)} \in \mathbf{X}^{(\mathtt{fin})}$ such that $d_H(I', I) < \varepsilon/2$ and $U^{(k)}_{\hat\beta_N,N}(I) \le U^{(k)}_{\hat\beta,\infty}(I') + \eta/4$ with probability larger than $1 - \delta/4$. This relation implies that $\hat{u}^{(k)}_{\hat\beta_N,N}(\varepsilon) \le \hat{u}^{(k)}_{\hat\beta,\infty}(\varepsilon/4) + \eta/4$, because $d_H(I', \hat{I}^{(k)}_{\hat\beta,\infty}) > \varepsilon/4$ and (4.59) follows. The existence of I' is explicit: we observe that $I = \{0, Y^{(N)}_{i_1}, \cdots, Y^{(N)}_{i_\ell}, 1\}$, for a suitable choice of indexes $\{i_1, \cdots, i_\ell\} \subset \{1, \cdots, k\}$. By using Lemma 4.21 it is not difficult to show that we can choose $I' = \{0, Y^{(\infty)}_{i_1}, \cdots, Y^{(\infty)}_{i_\ell}, 1\}$, possibly by enlarging N.

The proof of (4) follows by observing that by Part (1), there exists $\eta > 0$ and $K > 0$ such that $\hat{u}^{(k)}_{\hat\beta,\infty}(\varepsilon/4) \le \hat{u}^{(k)}_{\hat\beta,\infty} - \eta$ with probability larger than $1 - \delta/4$, for any $k > K$. This provides an upper bound for (4.59) and Part (2) allows to complete the proof. □

Let us stress that for any fixed $N \in \mathbb{N}$ we have that $\hat{I}^{(k)}_{\hat\beta_N,N} \equiv \hat{I}_{\hat\beta_N,N}$ as $k > N$. In the following Proposition we show that this convergence holds uniformly on N.

Proposition 4.25. *The following hold*

(1) *For any $N, k \in \mathbb{N} \cup \{\infty\}$ we define*

$$\rho_N^{(k)} := \sup_{I\in\mathbf{X}} \left|\hat\sigma_N(I) - \hat\sigma_N^{(k)}(I)\right| = \sum_{i>k} M_i^{(N)}. \tag{4.60}$$

Then for any $\varepsilon, \delta > 0$ there exists $K > 0$ such that $\mathbb{P}(\rho_N^{(k)} > \varepsilon) < \delta$ for all $k > K$, uniformly on $N \in \mathbb{N}$.

(2) $\mathbb{P}\left(\lim_{k\to\infty} \hat{I}^{(k)}_{\hat\beta,\infty} = \hat{I}_{\hat\beta,\infty}\right) = 1.$

(3) *For any $\varepsilon, \delta > 0$ there exists $K > 0$ such that $\mathbb{P}\left(d_H(\hat{I}^{(k)}_{\hat\beta_N,N}, \hat{I}_{\hat\beta_N,N}) < \varepsilon\right) > 1 - \delta$ for all $k > K$, uniformly on N.*

Proof. Part (1) is similar to [50, Proposition 3.3] and actually simpler. Here we give a short sketch of the proof. We note that if $k \ge N$, then $\rho_N^{(k)} \equiv 0$, therefore we can suppose $k < N$. For such k we consider the "good event", like in [50, (3.8)]

$$\mathcal{B}_k^{(N)} = \left\{ F^{-1}\left(1 - \frac{2r}{N}\right) \leq \tilde{M}_r^{(N)} \leq F^{-1}\left(1 - \frac{1}{N}\right), \text{ for all } k \leq r \leq N-1 \right\}. \tag{4.61}$$

Then, cf. [50, Lemma 3.4] $\mathbb{P}\left(\mathcal{B}_k^{(N)}\right) \to 1$ as $k \to \infty$, uniformly on N. By partitioning with respect to the "good event" and then by using Markov's inequality, we get that for any $\varepsilon > 0$

$$\mathbb{P}\left(\rho_N^{(k)} > \varepsilon\right) \leq \mathbb{P}\left(\mathcal{B}_k^{(N)} \text{ fails}\right) + \varepsilon^{-1} \sum_{r=k}^{N-1} \mathbb{E}\left[M_r^{(N)}; \mathcal{B}_k^{(N)}\right]. \tag{4.62}$$

To conclude the proof it is enough to show that $\sum_{r=k}^{N-1} \mathbb{E}\left[M_r^{(N)}; \mathcal{B}_k^{(N)}\right]$ converges to 0 as $k \to \infty$, uniformly on $N > k$. An upper bound for $\mathbb{E}\left[M_r^{(N)}; \mathcal{B}_k^{(N)}\right]$ is provided by [50, Lemma 3.8]: for any $\delta > 0$ there exist c_0, c_1 and $c_2 > 0$ such that for any $2(1 + 1/\alpha) < k < r < N$

$$\mathbb{E}\left[M_r^{(N)}; \mathcal{B}_k^{(N)}\right] \leq c_0 r^{-\frac{1}{\alpha}+\delta} + c_1 b_N^{-1} \mathbb{1}_{\{r > c_2 n\}}.$$

This allows to conclude that there exist $c_0', c_1' > 0$ such that

$$\sum_{r=k}^{N-1} \mathbb{E}\left[M_r^{(N)}; \mathcal{B}_k^{(N)}\right] \leq c_0' k^{-\frac{1}{\alpha}+1+\delta} + c_1' N b_N^{-1}.$$

Since $\alpha \in (0, 1)$ and $N b_N^{-1} \to 0$ as $N \to \infty$, cf. (4.16), we conclude that the r.h.s. converges to 0 as $k \to \infty$, uniformly on $N > k$. □

Part (2) has been already proven in the proof of Part (1) of Proposition 4.24. □

Part (3) is a consequence of (4). Let us fix k such that (4) holds for any $N > N_k$ and that $\mathbb{P}(\hat{\beta}_N \rho_N^{(k)} < \eta/4) > 1 - \delta$ uniformly on N, cf. (4.60). In such case we claim that, for any $\ell > k$ and $N > N_k$

$$d_H(\hat{I}^{(k)}_{\hat{\beta}_N,N}, \hat{I}^{(\ell)}_{\hat{\beta}_N,N}) < \varepsilon, \tag{4.63}$$

with probability larger than $1 - 2\delta$. Otherwise if $d_H(\hat{I}^{(k)}_{\hat{\beta}_N,N}, \hat{I}^{(\ell)}_{\hat{\beta}_N,N}) \geq \varepsilon$ for some $\ell > k$, then it holds that

$$\hat{u}^{(\ell)}_{\hat{\beta}_N,N} \leq U^{(k)}_{\hat{\beta}_N,N}(\hat{I}^{(\ell)}_{\hat{\beta}_N,N}) + \hat{\beta}_N \rho_N^{(k)} \leq \hat{u}^{(k)}_{\hat{\beta}_N,N}(\varepsilon) + \eta/4. \tag{4.64}$$

Relation (4) provides an upper bound for the r.h.s. of (4.64), giving $\hat{u}^{(\ell)}_{\hat{\beta}_N,N} \leq \hat{u}^{(k)}_{\hat{\beta}_N,N} - \eta/4$ and this is a contradiction because $\ell \mapsto \hat{u}^{(\ell)}_{\hat{\beta}_N,N}$ is non-decreasing and thus $\hat{u}^{(\ell)}_{\hat{\beta}_N,N} \geq \hat{u}^{(k)}_{\hat{\beta}_N,N}$. By using (4.63) together with the triangle inequality we conclude that for any $\ell > k$ and $N > N_k$

$$d_H(\hat{I}_{\hat{\beta}_N,N}, \hat{I}^{(\ell)}_{\hat{\beta}_N,N}) < 2\varepsilon, \tag{4.65}$$

with probability larger than $1 - 4\delta$. To conclude we have to consider the case in which $N \leq N_k$. For any such N, $\hat{I}^{(k)}_{\hat{\beta}_N,N}$ converges to $\hat{I}_{\hat{\beta}_N,N}$ as $k \to \infty$. To be more precise, whenever $k > N$ we have that $\hat{I}^{(k)}_{\hat{\beta},N} = \hat{I}_{\hat{\beta},N}$. This concludes the proof. □

4.3.2 Proof of theorem 4.5

The proof is a consequence of [13, Theorem 3.2], which can be written as follows

Theorem 4.26. *Let us suppose that the r.v's $X_N^{(k)}, X^{(k)}, X_N, X$ take values in a separable metric space*

$(\mathcal{S}, d_{\mathcal{S}})$ *and* $X_N^{(k)}, X_N$ *are defined on the same probability space. Then if the following diagram holds*

$$\begin{tikzcd} X_N^{(k)} \arrow[r, "(\mathrm{d})", "N\to\infty"'] \arrow[d, "k\to\infty", "\textit{in probability, uniformly in } N"'] & X^{(k)} \arrow[d, "k\to\infty", "(\mathrm{d})"'] \\ X_N & X \end{tikzcd}$$

then $X_N \xrightarrow{(\mathrm{d})} X$*. The expression* in probability, uniformly in N *means*

$$\lim_{k\to\infty} \limsup_{N\to\infty} \mathbb{P}\left(d_{\mathcal{S}}(X_N^{(k)}, X_N) \geq \varepsilon\right) = 0, \tag{4.66}$$

for any fixed $\varepsilon > 0$.

In our case we have $X_N^{(k)} = \hat{I}^{(k)}_{\hat{\beta}_N,N}$, $X^{(k)} = \hat{I}^{(k)}_{\hat{\beta},\infty}$, $X_N = \hat{I}_{\hat{\beta}_N,N}$ and $X = \hat{I}_{\hat{\beta},\infty}$ and by Propositions 4.24 and 4.25 the diagram above holds.

Remark 4.27. Let us stress that under the coupling introduced in Lemma 4.21 we have that in Theorem 4.5 the convergence of $\hat{I}_{\hat{\beta}_N,N}$ to $\hat{I}_{\hat{\beta},\infty}$ holds in probability, namely for any $\varepsilon, \delta > 0$ one has $\mathbb{P}\left(d_H(\hat{I}_{\hat{\beta}_N,N}, \hat{I}_{\hat{\beta},\infty}) < \varepsilon\right) > 1 - \delta$ for all N large enough. This follows by Part (3) of Proposition 4.24 and Parts (2), (3) of Proposition 4.25.

4.4 Concentration

In this section we discuss the concentration of $\tau/N \cap [0,1]$ around the set $\hat{I}_{\hat{\beta}_N,N}$, cf. (4.23), giving a proof of Theorems 4.4, 4.6.

4.4.1 General Setting of the Section

Let us stress that in the pinning model (4.1) we can replace $h > 0$ in the exponent of the Radon-Nikodym derivative by $h = 0$ by replacing the original renewal τ with a new one, $\tilde{\tau}$ defined by $\mathrm{P}(\tilde{\tau}_1 = n) = e^h \mathrm{P}(\tau_1 = n)$ and $\mathrm{P}(\tilde{\tau}_1 = \infty) = 1 - e^h$. Note that the renewal process $\tilde{\tau}$ is terminating because $h < 0$. In this case (cf. Appendix 4.A) the renewal function $\tilde{u}(n) := \mathrm{P}(n \in \tilde{\tau})$ satisfies

$$\lim_{n\to\infty} \frac{\log \tilde{u}(n)}{n^\gamma} = -\mathsf{c}, \tag{4.67}$$

with the same γ and c used in Assumptions 4.2 for the original renewal process τ.

In the sequel we assume $\mathsf{c} = 1$ (as already discussed in the Section 4.3), $h = 0$ and we omit the tilde-sign on the notations, writing simply τ and $u(\cdot)$ instead of $\tilde{\tau}$ and $\tilde{u}(\cdot)$.

4.4.2 Proof of Theorem 4.4

To prove Theorem 4.4 we proceed in two steps. In the first one we consider a truncated version of the Gibbs measure (4.1) in which we regard only the firsts k-maxima among $\omega_1, \cdots, \omega_{N-1}$ and we prove concentration for such truncated pinning model, cf. Lemma 4.30. In the second step we show how to deduce Theorem 4.4.

Let us define the truncated pinning model. For technical reasons it is useful to write the energy using $\hat{\sigma}_N$ defined in (4.20).

Definition 4.28. For $N, k \in \mathbb{N}, \beta > 0$, the k-truncated Pinning Model measure is a probability measure defined by the following Radon-Nikodym derivative

$$\frac{\mathrm{d}\hat{\mathrm{P}}^{(k)}_{\beta,N}}{\mathrm{dP}_N}(I) = \frac{e^{N^\gamma \beta \hat{\sigma}_N^{(k)}(I)} \mathbb{1}(1 \in I)}{\hat{Z}^{(k)}_{\beta,N}}, \tag{4.68}$$

where P_N is the law of $\tau/N \cap [0,1]$ used in (4.1).

In the sequel we use the convention that whenever $k \geq N$, the superscript (k) will be omitted.

Remark 4.29. Note that whenever $\beta = \hat{\beta}_N$, the Radon-Nikodym derivative (4.68) with $k \geq N$ recovers the original definition (4.1) with $\beta = \beta_N$.

Lemma 4.30. *Let $(\hat{\beta}_N)_N$ be a sequence converging to $\hat{\beta} \in (0,\infty)$. For any fixed $\varepsilon, \delta > 0$ there exist $\nu = \nu(\varepsilon,\delta) > 0$, $K = K(\varepsilon,\delta)$ and $(N_k)_{k\geq K}$ such that*

$$\mathbb{P}\left(\hat{\mathrm{P}}^{(k)}_{\hat{\beta}_N,N}\left(d_H(I,\hat{I}^{(k)}_{\hat{\beta}_N,N}) > \delta\right) \leq e^{-N^\gamma \nu}\right) > 1-\varepsilon \tag{4.69}$$

for all $k > K$ and $N > N_k$.

Roughly speaking to prove Lemma 4.30 we need to estimate the probability that a given set $\iota = \{\iota_1,\cdots,\iota_\ell\}$, with $\iota_j < \iota_{j+1}$, is contained in τ/N. In other words we need to compute the probability that $\iota_1,\cdots,\iota_\ell \in \tau/N$ when N is large enough.

For this purpose we fix $\iota = \{\iota_1,\cdots,\iota_\ell\} \subset [0,1]$ and we consider $\iota^{(N)} = \{\iota_1^{(N)},\cdots,\iota_\ell^{(N)}\}$, where $\iota_i^{(N)}$ is the nearest point to ι_i in the lattice $\{0,1/N,\cdots,1\}$. We define $u_N(\iota) = \prod_{i=1}^{\ell} u(N(\iota_i^{(N)} - \iota_{i-1}^{(N)}))$, with $\iota_0^{(N)} := 0$. The behavior of $u_N(\iota)$ as $N \to \infty$ is given by the following result

Proposition 4.31. *Let $\iota = \{\iota_1,\cdots,\iota_\ell\} \subset [0,1]$ be a fixed and finite set and consider the associated real sequence $(u_N(\iota))_N$. Then $\lim_{N\to\infty} \frac{1}{N^\gamma} \log u_N(\iota) = -\sum_{i=1}^{\ell}(\iota_i - \iota_{i-1})^\gamma$ and it holds uniformly in the space of all subsets ι with points spaced at least by ξ, for any fixed $\xi > 0$.*

Proof. The convergence for a fixed set is a consequence of (4.67). To prove the uniformity we note that if $\iota_i - \iota_{i-1} > \xi$, then $\iota_i^{(N)} - \iota_{i-1}^{(N)} > \xi/2$ as soon as $1/N < \xi/2$, which is independent of such ι. This shows the claim for all such ι with two points and this concludes the proof because $u_N(\iota)$ is given by at most $\frac{1}{\xi} + 1$-factors in this form. □

Another simple, but important, observation is that for a fixed $k \in \mathbb{N}$, with high probability the minimal distance between $Y_1^{(N)},\cdots,Y_k^{(N)}$ (the positions of the first k-maxima introduced in Section 4.2) cannot be too small even if N gets large. To be more precisely, by using Lemma 4.21, we have that for any fixed $\varepsilon > 0$ and $k \in \mathbb{N}$ there exists $\xi = \xi(k,\varepsilon) > 0$ and N_k such that for any $N > N_k$ the event

$$\left\{\left|Y_i^{(N)} - Y_j^{(N)}\right| > \xi,\ Y_\ell^{(N)} \in (\xi, 1-\xi),\ \forall\, \ell,\ i \neq j \in \{1,\cdots,k\}\right\} \tag{4.70}$$

has probability larger than $1-\varepsilon$. By Proposition 4.31 this implies that for any fixed $\zeta > 0$ on the event (4.70), for all N large enough and uniformly on $\iota = \{\iota_0 = 0 < \iota_1 < \cdots < \iota_\ell < 1 = \iota_{\ell+1}\} \subset Y^{(N,k)}$, cf. (4.26), it holds that

$$e^{-N^\gamma E(\iota) - \zeta N^\gamma} \leq \mathrm{P}(\iota_1,\cdots,\iota_\ell \in \tau/N) \leq e^{-N^\gamma E(\iota) + \zeta N^\gamma}, \tag{4.71}$$

where $E(\iota) = \sum_{i=1}^{\ell+1}(\iota_i - \iota_{i-1})^\gamma$ is the entropy of the set ι, cf. (4.29).

Proof of Lemma 4.30. The aim of this proof is to show that for any given $\delta > 0$ and $k \in \mathbb{N}$ large enough, $\hat{\mathrm{P}}^{(k)}_{\hat{\beta}_N,N}\left(d_H(\hat{I}^{(k)}_{\hat{\beta}_N,N}, I) > \delta\right) \to 0$ as $N \to \infty$, with an explicit rate of convergence. Our strategy is the following: given a set $I \subset \{0,1/N,\cdots,1\}$, with $0,1 \in I$, we consider

$$I_{(N,k)} := I \cap Y^{(N,k)}, \tag{4.72}$$

the intersection of I with the set of the positions of the firsts k-maxima: it can have distance larger or smaller than $\frac{\delta}{2}$ from $\hat{I}^{(k)}_{\hat{\beta}_N,N}$. This induces a partition of the set of all possible I's. This allows us to get the following inclusion of events

$$\left\{d_H(\hat{I}^{(k)}_{\hat{\beta}_N,N}, I) > \delta\right\} \subset \left\{d_H(\hat{I}^{(k)}_{\hat{\beta}_N,N}, I_{(N,k)}) \geq \frac{\delta}{2}\right\} \cup \left\{d_H(\hat{I}^{(k)}_{\hat{\beta}_N,N}, I_{(N,k)}) < \frac{\delta}{2}, d_H(I_{(N,k)}, I) > \frac{\delta}{2}\right\}. \tag{4.73}$$

We have thus to prove our statement for

$$\hat{\mathrm{P}}^{(k)}_{\hat{\beta}_N,N}\left(d_H(\hat{I}^{(k)}_{\hat{\beta}_N,N}, I_{(N,k)}) \geq \frac{\delta}{2}\right), \tag{4.74}$$

$$\hat{\mathrm{P}}^{(k)}_{\hat{\beta}_N,N}\left(d_H(\hat{I}^{(k)}_{\hat{\beta}_N,N}, I_{(N,k)}) < \frac{\delta}{2}, d_H(I, I_{(N,k)}) > \frac{\delta}{2}\right). \tag{4.75}$$

For this purpose we fix $\varepsilon > 0$ and $\xi = \xi(\varepsilon, k) > 0$, $N_k > 0$ such that the event (4.70) holds with probability larger than $1 - \varepsilon$, for any $N > N_k$.

Our goal is to find a good upper bound for (4.74) and (4.75). Let us start to consider (4.74). Let A be the set of all possible values of $I_{(N,k)}$, cf. (4.72), on the event $\{d_H(\hat{I}^{(k)}_{\hat{\beta}_N,N}, I_{(N,k)}) \geq \frac{\delta}{2}\}$, namely

$$A = \left\{\iota \subset Y^{(N,k)} : d_H(\iota, \hat{I}^{(k)}_{\hat{\beta}_N,N}) \geq \frac{\delta}{2} \text{ and } 0, 1 \in \iota\right\}. \tag{4.76}$$

An upper bound of (4.74)) is

$$\hat{\mathrm{P}}^{(k)}_{\hat{\beta}_N,N}\left(d_H(\hat{I}^{(k)}_{\hat{\beta}_N,N}, I_{(N,k)}) \geq \frac{\delta}{2}\right) \leq \sum_{\iota \in A} \hat{\mathrm{P}}^{(k)}_{\hat{\beta}_N,N}(I_{(N,k)} = \iota). \tag{4.77}$$

Let us fix $\zeta > 0$ (we choose in a while its precise value) and assume that Relation (4.71) holds if N_k is sufficiently large. Then

$$\hat{\mathrm{P}}^{(k)}_{\hat{\beta}_N,N}(I_{(N,k)} = \iota) = \frac{\mathrm{E}_N\left(e^{N^\gamma \hat{\beta}_N \hat{\sigma}^{(k)}_N(I)} \mathbb{1}(I_{(N,k)} = \iota); 1 \in \tau/N\right)}{\mathrm{E}_N\left(e^{N^\gamma \hat{\beta}_N \hat{\sigma}^{(k)}_N(I)}; 1 \in \tau/N\right)} \leq \frac{e^{N^\gamma \hat{\beta}_N \hat{\sigma}^{(k)}_N(\iota)} \mathrm{P}_N\left(\iota \subset I\right)}{e^{N^\gamma \hat{\beta}_N \hat{\sigma}^{(k)}_N\left(\hat{I}^{(k)}_{\hat{\beta}_N,N}\right)} \mathrm{P}_N\left(\hat{I}^{(k)}_{\hat{\beta}_N,N} \subset I\right)} \overset{(4.71)}{\leq} \tag{4.78}$$

$$\leq \exp\left\{-N^\gamma(\hat{u}^{(k)}_{\hat{\beta}_N,N} - U^{(k)}_{\hat{\beta}_N,N}(\iota)) + 2N^\gamma \zeta\right\} \leq \exp\left\{-N^\gamma(\hat{u}^{(k)}_{\hat{\beta}_N,N} - \hat{u}^{(k)}_{\hat{\beta}_N,N}(\delta/2)) + 2N^\gamma \zeta\right\},$$

where $U^{(k)}_{\hat{\beta}_N,N}$ has been introduced in Definition 4.19. By Proposition 4.24, Part (4), if k and N_k are taken large enough, it holds that $\hat{u}^{(k)}_{\hat{\beta}_N,N} - \hat{u}^{(k)}_{\hat{\beta}_N,N}(\delta/2) > \eta$, for some $\eta > 0$, with probability larger than $1 - \varepsilon$. We conclude that if ζ in (4.71) is chosen smaller than $\eta/4$, then the l.h.s. of (4.78) is bounded by $e^{-N^\gamma \frac{\eta}{2}}$, uniformly in $\iota \in A$. By observing that A has at most 2^k elements we conclude that

$$(4.74) \leq \sum_{\iota \in A} \hat{\mathrm{P}}^{(k)}_{\hat{\beta}_N,N}(I_{(N,k)} = \iota) \leq |A| e^{-N^\gamma \eta/2} \leq 2^k e^{-N^\gamma \eta/2}. \tag{4.79}$$

For (4.75) we use the same strategy: Let B be the set of all possible values of $I_{(N,k)}$, cf. (4.72), on the event $\{d_H(\hat{I}^{(k)}_{\hat{\beta}_N,N}, I_{(N,k)}) < \frac{\delta}{2}\}$,

$$B = \left\{\iota \subset Y^{(N,k)} : d_H(\iota, \hat{I}^{(k)}_{\hat{\beta}_N,N}) < \frac{\delta}{2} \text{ and } 0, 1 \in \iota\right\}, \tag{4.80}$$

Then

$$\hat{\mathrm{P}}^{(k)}_{\hat{\beta}_N,N}\left(d_H(\hat{I}^{(k)}_{\hat{\beta}_N,N}, I_{(N,k)}) < \frac{\delta}{2}, d_H(I, I_{(N,k)}) > \frac{\delta}{2}\right) \leq \sum_{\iota \in B} \hat{\mathrm{P}}^{(k)}_{\hat{\beta}_N,N}\left(d_H(\iota, I) > \frac{\delta}{2}, I_{(N,k)} = \iota\right). \tag{4.81}$$

Let us observe that for such a given ι

$$\hat{P}^{(k)}_{\hat{\beta}_N,N}\left(d_H(\iota,I) > \frac{\delta}{2}, I_{(N,k)} = \iota\right) = \frac{E_N\left(e^{N^\gamma \hat{\beta}_N \hat{\sigma}^{(k)}_N(I)} \mathbb{1}(d_H(\iota,I) > \frac{\delta}{2}, I_{(N,k)} = \iota); 1 \in I\right)}{E_N\left(e^{N^\gamma \hat{\beta}_N \hat{\sigma}^{(k)}_N(I)}; 1 \in I\right)} \tag{4.82}$$
$$\leq \frac{P_N\left(d_H(\iota,I) > \frac{\delta}{2}, I_{(N,k)} = \iota\right)}{P_N(\iota \subset I)}.$$

We have reduced our problem to compute the probability of the event $\left\{d_H(\iota,I) > \frac{\delta}{2}, I_{(N,k)} = \iota\right\}$ under the original renewal distribution P_N.

Note that, if $\iota \subset I$, then $d_H(\iota,I) > \frac{\delta}{2}$ if and only if there exists $x \in I$ such that $d(x,\iota) > \frac{\delta}{2}$. Thus

$$\left\{d_H(\iota,I) > \frac{\delta}{2}, I_{(N,k)} = \iota\right\} = \left\{\exists x \in I, d(x,\iota) > \frac{\delta}{2}, I_{(N,k)} = \iota\right\}. \tag{4.83}$$

For $\iota = \{\iota_0 = 0 < \iota_1 < \cdots < \iota_\ell = 1\} \in B$, we define $U_{j,\delta} := [\iota_j + \frac{\delta}{2}, \iota_{j+1} - \frac{\delta}{2}] \cap \frac{\mathbb{N}}{N}$, which is empty if the distance between ι_j and ι_{j+1} is strictly smaller than δ. We can decompose the event (4.83) by using such $U_{j,\delta}$, i.e., $\{\exists x \in I, d(x,\iota) > \frac{\delta}{2}, I_{(N,k)} = \iota\} = \bigcup_{j=0}^{\ell-1} \bigcup_{x \in U_{j,\delta}} \{x \in I, I_{(N,k)} = \iota\}$, and we get

$$P_N(d_H(\iota,I) > \delta, I_{(N,k)} = \iota) \leq \sum_{j=0}^{\ell-1} \sum_{x \in U_{j,\delta}} P_N(x \in I, I_{(N,k)} = \iota) \leq \sum_{j=0}^{\ell-1} \sum_{x \in U_{j,\delta}} P_N(x \in I, \iota \subset I). \tag{4.84}$$

Let us consider $P_N(x \in I, \iota \subset I)$. Since x does not belong to ι, there exits an index j such that $\iota_j < x < \iota_{j+1}$. Then , recalling that $u(n) = P(n \in \tau)$,

$$\frac{P_N(x \in I, \iota \subset I)}{P_N(\iota \subset I)} = \frac{\left[\prod_{\substack{k=1,\\ k\neq j}}^{\ell-1} u(N(\iota_{k+1} - \iota_k))\right] u(N(x-\iota_j))u(N(\iota_{j+1}-x))}{\prod_{k=1}^{\ell-1} u(N(\iota_{k+1}-\iota_k))} = \frac{u(N(x-\iota_j))u(N(\iota_{j+1}-x))}{u(N(\iota_{j+1}-\iota_j))} \tag{4.85}$$
$$\overset{(4.71)}{\leq} e^{-N^\gamma\left((x-\iota_j)^\gamma + (\iota_{j+1}-x)^\gamma - (\iota_{j+1}-\iota_j)^\gamma\right) + 2\zeta N^\gamma} \leq e^{-N^\gamma(2^{1-\gamma}-1)\delta^\gamma + 2\zeta N^\gamma},$$

uniformly on all such ι_j, ι_{j+1} and x. Note that the last inequality follows by observing that for all such ι_j, ι_{j+1} and x one has $(x-\iota_j)^\gamma + (\iota_{j+1}-x)^\gamma - (\iota_{j+1}-\iota_j)^\gamma) \geq (2^{1-\gamma}-1)\delta^\gamma$. We conclude that, making possibly further restrictions on the value of ζ as function of δ, there exists a constant $C > 0$ such that $\frac{P_N(\{x\} \cup \iota \subset I)}{P_N(\iota \subset I)} \leq e^{-CN^\gamma}$ uniformly in $\iota \in B$. This leads to have that

$$(4.75) \leq \sum_{\iota \in B} \hat{P}^{(k)}_{\hat{\beta}_N,N}\left(d_H\left(\hat{I}^{(k)}_{\hat{\beta}_N,N}, I\right) > \frac{\delta}{2}, I_{(N,k)} = \iota\right) \leq |B| N e^{-CN^\gamma} \leq 2^k N e^{-CN^\gamma}. \tag{4.86}$$

□

Proof of Theorem 4.4. First of all we are going to prove concentration around $\hat{I}^{(k)}_{\hat{\beta}_N,N}$. Let $k > 0$ be fixed. Its precise value will be chosen in the following. Then, recalling Definition 4.28,

$$\hat{P}_{\hat{\beta}_N,N}\left(d_H\left(\hat{I}^{(k)}_{\hat{\beta}_N,N}, I\right) > \delta\right) \leq \tag{4.87}$$
$$\leq \hat{P}^{(k)}_{\hat{\beta}_N,N}\left(d_H\left(\hat{I}^{(k)}_{\hat{\beta}_N,N}, I\right) > \delta\right) \cdot \sup\left\{\frac{d\hat{P}_{\hat{\beta}_N,N}}{d\hat{P}^{(k)}_{\hat{\beta}_N,N}}(I) : d_H\left(\hat{I}^{(k)}_{\hat{\beta}_N,N}, I\right) > \delta\right\}.$$

To control the first term, by Lemma 4.30 for any $\varepsilon, \delta > 0$ there exists $\nu > 0$ and N_k such that for all $N > N_k$, $\hat{\mathrm{P}}^{(k)}_{\hat{\beta}_N,N}\left(d_H\left(\hat{I}^{(k)}_{\hat{\beta}_N,N}, I\right) > \delta\right) \leq e^{-N^\gamma \nu}$ with probability larger than $1-\varepsilon$. To control the Radon-Nikodym derivative we may write

$$\frac{\mathrm{d}\hat{\mathrm{P}}_{\hat{\beta}_N,N}}{\mathrm{d}\hat{\mathrm{P}}^{(k)}_{\hat{\beta}_N,N}}(I) = \frac{\hat{\mathrm{Z}}^{(k)}_{\hat{\beta}_N,N}}{\hat{\mathrm{Z}}_{\hat{\beta}_N,N}} \frac{e^{N^\gamma \hat{\beta}_N \hat{\sigma}_N(I)}}{e^{N^\gamma \hat{\beta}_N \hat{\sigma}^{(k)}_N(I)}} \leq e^{\hat{\beta}_N N^\gamma (\hat{\beta}_N \hat{\sigma}_N(I) - \hat{\beta}_N \hat{\sigma}^{(k)}_N(I))} \leq e^{\hat{\beta}_N N^\gamma \rho^{(k)}_N}, \tag{4.88}$$

where $\rho^{(k)}_N = \sum_{i>k} M^{(N)}_i$ is defined in (4.60). By using Part (1) of Proposition 4.25 we choose k large enough such that $\hat{\beta}_N \rho^{(k)}_N < \nu/2$ with probability $1-\varepsilon$, uniformly in N. This forces to have

$$\mathbb{P}\left(\hat{\mathrm{P}}_{\hat{\beta}_N,N}\left(d_H\left(\hat{I}^{(k)}_{\hat{\beta}_N,N}, I\right) > \delta\right) \leq e^{-N^\gamma \nu/2}\right) \geq 1 - 2\varepsilon. \tag{4.89}$$

The proof follows by observing that if k is large enough, then $d_H(\hat{I}^{(k)}_{\hat{\beta}_N,N}, \hat{I}_{\hat{\beta}_N,N}) < \delta/2$ with probability larger than $1-\varepsilon$, uniformly on N, cf. Point (3) Proposition 4.25. □

4.4.3 Proof of Theorem 4.6

In this section we prove Theorem 4.6. The proof is based on the following result

Lemma 4.32. *Let $(\mathcal{S}, d_{\mathcal{S}})$ be a metric space and let x_N be a sequence converging to $\bar{x}$. Let $\mu_N \in \mathcal{M}_1(\mathcal{S})$ be such that for any $\varepsilon > 0$, $\lim_{N\to\infty} \mu_N(x : d(x_N, x) > \varepsilon) = 0$. Then $\mu_N \rightharpoonup \delta_{\bar{x}}$.*

Proof. The proof is a consequence of the Portmanteau's Lemma [13, Section 2]. □

Proof of Theorem 4.6. Let $\mu_N = \hat{\mathrm{P}}_{\hat{\beta}_N,N}$ and $\mu_\infty = \delta_{\hat{I}_{\beta,\infty}}$. Note that μ_N is a random measure on $\mathbf{X}$ depending on the discrete disorder $w^{(N)}$, while μ_∞ depends on the continuum disorder $w^{(\infty)}$. Therefore if we couple together these disorders as in Lemma 4.21 we have that by Theorems 4.4, 4.5 (see Remark 4.27) $\mu_N\left(I \mid d_H(\hat{I}_{\hat{\beta}_N,N}, I) > \delta\right) \xrightarrow{\mathbb{P}} 0$ and $\hat{I}_{\hat{\beta}_N,N} \xrightarrow{\mathbb{P}} \hat{I}_{\beta,\infty}$. To conclude the proof let us observe that the law of μ_N is a probability measure on $\mathcal{M}_1(\mathbf{X})$, the space of the probability measures on $\mathbf{X}$, which is a compact space because $\mathbf{X}$ is compact. Therefore we can assume that μ_N has a limit in distribution. We have thus to show that this limit is the law of μ_∞. For this purpose it is enough to show that there exists a subsequence N_k such that $\mu_{N_k} \xrightarrow{(\mathrm{d})} \mu_\infty$. It is not difficult to check that we can find a subsequence N_k such that $\mu_{N_k}\left(I \mid d_H(\hat{I}_{\hat{\beta}_{N_k},N_k}, I) > \delta\right) \xrightarrow{\mathbb{P}-\text{a.s.}} 0$ and $\hat{I}_{\hat{\beta}_{N_k},N_k} \xrightarrow{\mathbb{P}-\text{a.s.}} \hat{I}_{\beta,\infty}$, therefore by Lemma 4.32 we conclude that $\mu_{N_k} \overset{\mathbb{P}-\text{a.s.}}{\rightharpoonup} \mu_\infty$ and this concludes the proof. □

4.5 Proof of Theorem 4.7

The goal of this section is to give a proof of Theorem 4.7.

As a preliminary fact let us show that if $\beta < \hat{\beta}_c$ then $\hat{I}_{\beta,\infty} \equiv \{0,1\}$, while if $\beta > \hat{\beta}_c$ then $\hat{I}_{\beta,\infty} \not\equiv \{0,1\}$.

To this aim let us consider the maximum of the difference between the Continuum Energy (4.24) and the entropy (4.30), $\hat{u}_{\beta,\infty} = \hat{\sigma}_\infty(\hat{I}_{\beta,\infty}) - E(\hat{I}_{\beta,\infty})$, defined in (4.36). Then whenever $\hat{u}_{\beta,\infty} \leq -1$, we have that $-1 = -E(\{0,1\}) \leq \hat{u}_{\beta,\infty} \leq -1$ and this implies that $\hat{I}_{\beta,\infty} \equiv \{0,1\}$ by uniqueness of the maximizer. On the other hand, if $\hat{u}_{\beta,\infty} > -1$, then there exists $I \not\equiv \{0,1\}$ such that $U_{\beta,\infty}(I) > -1$ because $U_{\beta,\infty}(\{0,1\}) = -1$, so that $\{0,1\} \subsetneq \hat{I}_{\beta,\infty}$. In particular, since $\beta \mapsto \hat{u}_{\beta,\infty}$ is non-decreasing, we have that $\hat{I}_{\beta,\infty} \equiv \{0,1\}$ if $\beta < \hat{\beta}_c$ and $\hat{I}_{\beta,\infty} \not\equiv \{0,1\}$ if $\beta > \hat{\beta}_c$.

To prove the theorem we proceed in two steps: in the first one we show that a.s. for any $\varepsilon > 0$ there exists $\beta_0 = \beta_0(\varepsilon) > 0$ random for which $\hat{I}_{\beta,\infty} \subset [0,\varepsilon] \cup [1-\varepsilon, 1]$ for all $\beta < \beta_0$. In the second one we show that if ε is small enough, then the quantity of energy that we can gain is always too small to hope to compensate the entropy. To improve this strategy we use some results on the Poisson Point Process that we are going to recall.

Let us start to note that the process $(Y_i^{(\infty)}, M_i^{(\infty)})_{i\in\mathbb{N}} \subset [0,1]\times\mathbb{R}_+$ is a realization of a Poisson Point Process Π with intensity

$$\mu(dxdz) = \mathbf{1}_{[0,1]}(x)\frac{\alpha}{z^{1+\alpha}}\mathbf{1}_{[0,\infty)}(z)dxdz. \tag{4.90}$$

In such a way, as proved in [59], the process

$$X_t = \sum_{(x,z)\in\Pi} z\mathbb{1}\,(x \in [0,t]\cup[1-t,1])\,, \quad t \in \left[0, \frac{1}{2}\right] \tag{4.91}$$

is a α-stable subordinator. The behavior of a α-stable subordinator in a neighborhood of 0 is described by [11, Thm 10 Ch. 3], precisely if $(X_t)_t$ is such subordinator with $\alpha \in (0,1)$ and $h : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing function, then $\limsup_{t\to 0^+} X_t/h(t) = \infty$ or 0 a.s. depending on whether the integral $\int_0^1 h(t)^{-\alpha}dt$ diverges or converges. In particular by taking $q > 1$ and $h(t) = t^{1/\alpha}\log^{q/\alpha}(1/t)$ in a neighborhood of 0, we have the following result

Proposition 4.33. *Let $(X_t)_t$ be a α-stable subordinator, with $\alpha \in (0,1)$, then for every $q > 1$ a.s. there exists a random constant $C > 0$ such that*

$$X_t \le Ct^{\frac{1}{\alpha}}\log^{\frac{q}{\alpha}}\left(\frac{1}{t}\right) \tag{4.92}$$

in a neighborhood of 0.

Remark 4.34. The process X_t is the value of the sum of all charges in the set $[0,t]\cup[1-t,1]$. Therefore it gives an upper bound on the energy that we can gain by visiting this set.

Step One.

Let us show that a.s. for any $\varepsilon > 0$ there exists $\beta_0 = \beta_0(\varepsilon) > 0$ for which $\hat{I}_{\beta,\infty} \subset [0,\varepsilon]\cup[1-\varepsilon,1]$ for all $\beta < \beta_0$. Otherwise there should exist $\varepsilon > 0$ and a sequence $\beta_k > 0$, $\beta_k \to 0$ as $k \to \infty$ such that $\hat{I}_{\beta_k,\infty} \cap (\varepsilon, 1-\varepsilon) \neq \emptyset$. Let x be one of such points, then, by Theorem 4.15 we have that $E(\hat{I}_{\beta_k,\infty}) \ge E(\{0,x,1\}) \ge \varepsilon^\gamma + (1-\varepsilon)^\gamma$. Let $S = \sum_{i\in\mathbb{N}} M_i^{(\infty)}$, which is a.s. finite, cf. Remark 4.11. Therefore by observing that $\hat{u}_{\beta_k,\infty} = \beta_k\hat{\sigma}_\infty(\hat{I}_{\beta_k,\infty}) - E(\hat{I}_{\beta_k,\infty}) \ge -1$ we get

$$\beta_k S \ge \beta_k\hat{\sigma}_\infty(\hat{I}_{\beta_k,\infty}) \ge E(\hat{I}_{\beta_k,\infty}) - 1 \ge \varepsilon^\gamma + (1-\varepsilon)^\gamma - 1. \tag{4.93}$$

There is a contradiction because the l.h.s. goes to 0 as $\beta_k \to 0$, while the r.h.s. is a strictly positive number.

Remark 4.35. Let us note that if we set $\beta_0 = \beta_0(\varepsilon) = (\varepsilon^\gamma + (1-\varepsilon)^\gamma - 1)/S$ then for all $\beta < \beta_0$ it must be that $\hat{I}_{\beta,\infty} \subset [0,\varepsilon]\cup[1-\varepsilon,1]$. Moreover $\beta_0 \downarrow 0$ as $\varepsilon \downarrow 0$.

Step Two.

Now let us fix $\varepsilon > 0$ small and $\beta_0 = \beta_0(\varepsilon) \le 1$ as in Remark 4.35. Let

$$\varepsilon_1 = \sup \hat{I}_{\beta,\infty} \cap [0,\varepsilon], \tag{4.94}$$

$$\varepsilon_2 = \inf \hat{I}_{\beta,\infty} \cap [1-\varepsilon, 1]. \tag{4.95}$$

Let $\hat{\varepsilon} = \max\{\varepsilon_1, 1-\varepsilon_2\}$. If $\hat{\varepsilon} = 0$ we have finished. Then we may assume that $\hat{\varepsilon} > 0$ and we choose $q > 1$, $C > 0$ for which Proposition 4.33 holds for any $t < \varepsilon$, namely

$$\beta\hat{\sigma}_\infty(\hat{I}_{\beta,\infty}) \le \beta_0 X_{\hat{\varepsilon}} \le C\hat{\varepsilon}^{\frac{1}{\alpha}} \log^{\frac{q}{\alpha}}\left(\frac{1}{\hat{\varepsilon}}\right), \tag{4.96}$$

By Theorem 4.15 we get a lower bound for the entropy

$$E(\hat{I}_{\beta,\infty}) \ge \hat{\varepsilon}^\gamma + (1-\hat{\varepsilon})^\gamma. \tag{4.97}$$

In particular if ε is small enough, we have that

$$E(\hat{I}_{\beta,\infty}) - 1 > \frac{\hat{\varepsilon}^\gamma}{2}. \tag{4.98}$$

Therefore with a further restrictions on ε and β_0, if necessary, by recalling that $\alpha, \gamma \in (0,1)$ we conclude that for all $\beta < \beta_0$

$$E(\hat{I}_{\beta,\infty}) - 1 \le \beta\hat{\sigma}_\infty(\hat{I}_{\beta,\infty}) \le C\hat{\varepsilon}^{\frac{1}{\alpha}} \log^{\frac{q}{\alpha}}\left(\frac{1}{\hat{\varepsilon}}\right) \le \frac{\hat{\varepsilon}^\gamma}{2} < E(\hat{I}_{\beta,\infty}) - 1, \tag{4.99}$$

which is a contradiction. Therefore $\hat{\varepsilon}$ must be 0 and this implies that $\hat{I}_{\beta,\infty} \equiv \{0,1\}$ for each $\beta < \beta_0$.

4.6 The Directed polymer in random environment with heavy tails

Originally introduced by [52], the directed polymer in random environment is a model to describe an interaction between a polymer chain and a medium with microscopic impurities. From a mathematical point of view we consider the set of all possible paths of a $1+1$ - dimensional simple random walk starting from 0 and constrained to come back to 0 after N-steps. The impurities — and so the medium-polymer interactions — are idealized by an i.i.d. sequence $(\{\omega_{i,j}\}_{i\in\mathbb{N}, j\in\mathbb{Z}}, \mathbb{P})$. Each random variable $\omega_{i,j}$ is placed on the point $(i,j) \in \mathbb{N}\times\mathbb{Z}$. For a given path s we define the Gibbs measure

$$\mu_{\beta,N}(s) = \frac{e^{\beta\sigma_N(s)}}{Q_{\beta,N}}, \tag{4.100}$$

where $\sigma_N(\cdot) = \sum_{i,j}\omega_{i,j}\mathbb{1}(s_i = j)$ is the energy and $Q_{\beta,N}$ is a normalization constant.

In [7] is studied the case in which the impurities have heavy tails, namely the distribution of $\omega_{1,1}$ is regularly varying with index $\alpha \in (0,2)$. In this case to have a non-trivial limit as $N \to \infty$, we have to choose $\beta = \beta_N \sim \hat{\beta}N^{1-2/\alpha}L(N)$, with L a slowly varying function, cf. [7, (2.4),(2.5)]. For such a choice of β, cf. [7, Theorem 2.1], one has that the trajectories of the polymer are concentrated in the uniform topology around a favorable curve $\hat{\gamma}_{\beta_N,N}$. In [7, Theorem 2.2] one shows that there exists a limit in distribution for the sequence of curves $\hat{\gamma}_{\beta_N,N}$, denoted by $\hat{\gamma}_\beta$. Moreover there exits a random threshold β_c below which such limit is trivial ($\hat{\gamma}_\beta \equiv 0$), cf. [7, Proposition 2.5]. Anyway a complete description of β_c it was not given, see Remark 4.9. In our work we solve this problem, cf. Theorem 4.8.

The rest of the section is consecrated to prove Theorem 4.8.

Definition 4.36 (entropy). Let us consider $\mathcal{L}^0 = \{s : [0,1] \to \mathbb{R} : s \text{ is } 1-\text{Lipschitz}, s(0) = s(1) = 0\}$ equipped with L^∞-norm, denoted by $\|\cdot\|_\infty$.

For a curve $\gamma \in \mathcal{L}^0$ we define its entropy as

$$E(\gamma) = \int_0^1 e\left(\frac{d}{dx}\gamma(x)\right) dx, \tag{4.101}$$

where $e(x) = \frac{1}{2}((1+x)\log(1+x) + (1-x)\log(1-x))$.

Let us observe that $E(\cdot)$ is the rate function in the large deviations principle for the sequence of uniform measures on $\mathcal{L}_N^0$, the set of linearly interpolated $\frac{1}{N}$-scaled trajectories of a simple random walk.

Definition 4.37. We introduce the continuous environment π_∞ as

$$\pi_\infty(\gamma) = \sum_i T_i^{-\frac{1}{\alpha}} \delta_{Z_i}(\mathrm{graph}(\gamma)), \gamma \in \mathcal{L}^0. \tag{4.102}$$

Here $\mathrm{graph}(\gamma) = \{(x, \gamma(x)) : x \in [0,1])\} \subset \mathcal{D} := \{(x,y) \subset \mathbb{R}^2 : |y| \le x \wedge (1-x)\}$ is the graph of γ, $\alpha \in (0,2)$ is the parameter related to the disorder, T_i is a sum of i-independent exponentials of mean 1 and $(Z_i)_{i\in\mathbb{N}}$ is an i.i.d.-sequence of Uniform($\mathcal{D}$) r.v.'s. These two sequences are assumed to be independent with joint law denoted by $\mathbb{P}_\infty$.

For $\beta < \infty$ we introduce

$$\hat{\gamma}_\beta = \arg\max_{\gamma\in\mathcal{L}^0} \{\beta\pi_\infty(\gamma) - E(\gamma)\} \tag{4.103}$$

and we set $u_\beta = \beta\pi_\infty(\hat{\gamma}_\beta) - E(\hat{\gamma}_\beta)$. Since $\beta\pi_\infty(\gamma \equiv 0) - E(\gamma \equiv 0) = 0$ a.s. we have that $u_\beta \ge 0$ a.s., consequently we define the random threshold as

$$\beta_c = \inf\{\beta > 0 : u_\beta > 0\} = \inf\{\beta > 0 : \hat{\gamma}_\beta \not\equiv 0\}. \tag{4.104}$$

4.6.1 The Structure of β_c

The random set $(Z_i, T_i^{-\frac{1}{\alpha}})_{i\in\mathbb{N}} \subset \mathcal{D} \times \mathbb{R}_+$ is a realization of a Poisson Point Process , denoted by Π^*, with density given by

$$\mu^*(dxdydz) = \frac{\mathbf{1}_{\mathcal{D}}(x,y)}{|\mathcal{D}|} \frac{\alpha}{z^{1+\alpha}} \mathbf{1}_{[0,\infty)}(z) dxdydz. \tag{4.105}$$

Let us introduce the process

$$U_t = \sum_{(x,y,z)\in\Pi^*} z\mathbb{1}((x,y) \in A(t)), \quad t \in \left[0, \frac{1}{2}\right] \tag{4.106}$$

with $A(t) = \{(x,y) \in \mathcal{D} : x \in [0,1], |y| \le t\}$. Let us observe that the process $(U_t)_{t\in[0,\frac{1}{2}]}$ is "almost" a Lévy Process, in sense that it has càdlàg trajectories and independent but not homogeneous increments because the area of $A(t)$ does not grow linearly. Anyway, by introducing a suitable function $\phi(t) > t$, we can replace $A(t)$ by $A(\phi(t))$ to obtain a process with homogeneous increment. In particular we take $\phi(t) = 1/2(1 - \sqrt{1-4t})$ in order to have that $\mathrm{Leb}(A(\phi(t))) = t$ for all $t \in [0, 1/4]$. Then the process

$$W_t = U_{\phi(t)} \tag{4.107}$$

is a subordinator and $W_t \ge U_t$ for any $t \in [0, 1/4]$.

Before giving the proof of Theorem 4.8, we prove a general property of the model:

Proposition 4.38. *For any fixed $\alpha \in (0,2)$, $\mathbb{P}_\infty$-a.s. for any $\varepsilon > 0$ there exists $\beta_0 = \beta_0(\varepsilon) > 0$ such that $\|\hat{\gamma}_\beta\|_\infty < \varepsilon$ (that is, $\mathrm{graph}(\hat{\gamma}_\beta) \subset A(\varepsilon)$) for all $\beta < \beta_0$.*

Let us recall some preliminary results necessary for the proof.

Proposition 4.39. *Let E be the entropy of Definition 4.36. Then for all $\gamma \in \mathcal{L}^0$ if $z = (x, y) \in graph(\hat{\gamma}_\beta)$ we have that*

$$E(\gamma) \geq E(\gamma_z), \tag{4.108}$$

where γ_z is the curve obtained by linear interpolation of $\{(0,0), z, (1,0)\}$.

Proof. [7, Proposition 3.1] □

As shown in [7, Proof of Proposition 2.5], there exist two constants $C_1, C_2 > 0$ such that for all $z = (x, y) \in \mathcal{D}$ we have $C_1 \left(\frac{y^2}{x} + \frac{y^2}{1-x}\right) \leq E(\gamma_z) \leq C_2 \left(\frac{y^2}{x} + \frac{y^2}{1-x}\right)$. This implies that there exists $C_0 > 0$ for which

$$E(\gamma_z) \geq C_0 y^2, \tag{4.109}$$

uniformly on $z \in \mathcal{D}$.

Proof of Proposition 4.38. By contradiction let us suppose that there exists $\varepsilon > 0$ such that for a sequence $\beta_k \to 0$ as $k \to \infty$ we have $\|\hat{\gamma}_{\beta_k}\|_\infty \geq \varepsilon$. By continuity of $\hat{\gamma}_{\beta_k}$ there exists a point $x \in [\varepsilon, 1-\varepsilon]$ such that $\hat{\gamma}_{\beta_k}(x) = \varepsilon$. By [50, Proposition 4.1], with probability 1 there exists a random set $A \subset \mathbb{N}$ such that $S = \sum_{i \in A} T_i^{-\frac{1}{\alpha}} < \infty$ and for any $\gamma \in \mathcal{L}^0$ it holds that $S \geq \pi_\infty(\gamma)$. For instance if $\alpha \in (0,1)$, then we can choose $A \equiv \mathbb{N}$, while if $\alpha > 1$, then $A \subsetneq \mathbb{N}$. Since $u_{\beta_k} = \beta_k \pi_\infty(\hat{\gamma}_{\beta_k}) - E(\hat{\gamma}_{\beta_k}) \geq 0$ we obtain that a.s.

$$\beta_k S \geq \beta_k \pi_\infty(\hat{\gamma}_{\beta_k}) \geq E(\hat{\gamma}_{\beta_k}) \geq E(\gamma_{z=(x,\varepsilon)}) \geq C_0 \varepsilon^2. \tag{4.110}$$

Sending $\beta_k \to 0$ we obtain a contradiction because the l.h.s. converges to 0. □

We are now ready to prove Theorem 4.8.

Proof of Theorem 4.8. We have to prove only the point (1), the other one has been already proven in [7]. Let $\varepsilon > 0$ be fixed and $\beta_0 = \beta_0(\varepsilon)$ such that $\|\hat{\gamma}_\beta\|_\infty < \varepsilon$ for all $\beta < \beta_0 \leq 1$. Moreover we define $\hat{\varepsilon} := \max |\hat{\gamma}_\beta(x)|$.

An upper bound for the energy gained by $\hat{\gamma}_\beta$ is given by $\sum_{i \in \mathbb{N}} T_i^{-\frac{1}{\alpha}} \mathbf{1}_{(Z_i \in A_{\hat{\varepsilon}})}$, the sum of all charges contained in the region $A_{\hat{\varepsilon}}$. Such quantity is estimated by the process $W_{\hat{\varepsilon}}$, cf. (4.107). Therefore by Proposition 4.33 we can choose suitable constants $q > 1$ and $C > 0$ such that

$$\pi_\infty(\hat{\gamma}_\beta) \leq \sum_{i \in \mathbb{N}} T_i^{-\frac{1}{\alpha}} \mathbf{1}_{(Z_i \in A_{\hat{\varepsilon}})} \leq U_{\hat{\varepsilon}} \leq C \hat{\varepsilon}^{\frac{1}{\alpha}} \log^{\frac{q}{\alpha}} \left(\frac{1}{\hat{\varepsilon}}\right). \tag{4.111}$$

A lower bound for the entropy is provided by (4.109):

$$E(\hat{\gamma}_\beta) \geq C_0 \hat{\varepsilon}^2. \tag{4.112}$$

Conclusion: if ε is small enough we get

$$\beta \pi_\infty(\hat{\gamma}_\beta) \leq C \hat{\varepsilon}^{\frac{1}{\alpha}} \log^{\frac{q}{\alpha}} \left(\frac{1}{\hat{\varepsilon}}\right) \leq C_0 \hat{\varepsilon}^2 \leq E(\hat{\gamma}_\beta), \tag{4.113}$$

because $\alpha < \frac{1}{2}$ and this forces $u_\beta = 0$ for all $\beta < \beta_0$. □

4.A Asymptotic Behavior for Terminating Renewal Processes

In this section we consider a terminating renewal process (τ, P) and $K(n) = \mathrm{P}(\tau_1 = n)$, with $K(\infty) > 0$. The aim is to study the asymptotic behavior of the renewal function $u(N) = \mathrm{P}(N \in \tau) = \sum_m K^{*(m)}(N)$, where $K^{*(m)}$ is the m^{th}-convolution of K with itself, under the assumption that $K(\cdot)$ is subexponential. We refer to [41] for the general theory of the subexponential distributions.

Definition 4.40 (Subexponential distribution)**.** We say that a discrete probability density q on $\mathbb{N}$ is subexponential if

$$\forall\, k > 0,\ \lim_{n\to\infty} q(n+k)/q(n) = 1 \quad \text{and} \quad \lim_{n\to\infty} q^{*(2)}(n)/q(n) = 2, \tag{4.114}$$

The result we are interested in is the following

Theorem 4.41. *Let $K(\cdot)$ be a discrete probability density on $\mathbb{N}\cup\{\infty\}$ such that $K(\infty) > 0$ and let $\delta = 1 - K(\infty) < 1$. Let $q(\cdot)$ defined as $q(n) = \delta^{-1}K(n)$. If q is subexponential, then*

$$\lim_{n\to\infty} \frac{u(n)}{K(n)} = \frac{1}{K(\infty)^2}. \tag{4.115}$$

Its proof is a simple consequence of the Dominated Convergence Theorem by using the following results

Lemma 4.42. *Let q be a subexponential discrete probability density on $\mathbb{N}$, then for any $m \geq 1$*

$$q^{*(m)}(n) \overset{n\to\infty}{\sim} mq(n). \tag{4.116}$$

Proof. [41, Corollary 4.13]. □

Theorem 4.43. *Let q be a subexponential discrete probability density on $\mathbb{N}$. Then we have that for any $\varepsilon > 0$ there exists $N_0 = N_0(\varepsilon)$ and $c = c(\varepsilon)$ such that for any $n > N_0$ and $m \geq 1$*

$$q^{*(m)}(n) \leq c(1+\varepsilon)^m q(n). \tag{4.117}$$

Proof. [41, Theorem 4.14]. □

4.A.1 The case of $K(n) \cong e^{-\mathsf{c}n^\gamma}$.

In this section we want to show that (4.4) satisfies Assumption 4.2. The fact that it is stretched-exponential, (2), is obvious, then it is left to prove that it is subexponential, (1).

By [41, Theorem 4.11], we can assume $K(n) = n^\rho \tilde{L}(n)e^{-\mathsf{c}n^\gamma}$, where $\tilde{L}$ is another slowly varying function such that $\tilde{L}(n) \sim L(n)$ as $n \to \infty$. Since $\gamma \in (0,1)$ we get that for any fixed $k > 0$, $\lim_{n\to\infty} K(n+k)/K(n) = 1$. Such property goes under the name of long-tailed and it allows to apply [41, Theorem 4.7]: to prove that K is subexponential, we have to prove that for any choice of $h = h(n) \to \infty$ as $n \to \infty$, with $h(n) < n/2$, we have that $\sum_{m=h(n)}^{n-h(n)} K(n-m)K(m) = o(K(n))$, as $n \to \infty$. Let us consider $R(y) = y^\gamma$, with $\gamma \in (0,1)$. R is a concave increasing function and $R'(y) = \gamma y^{\gamma-1}$ is strictly decreasing, so that given two integer points n, m such that $n - m > m$ we have

$$R(n) - R(n-m) \leq mR'(n-m) \leq mR'(m) = \gamma m^\gamma = \gamma R(m), \tag{4.118}$$

By Karamata's representation for slowly varying functions [14, Theorem 1.2.1] there exists $c_1 \geq 1$ for which $\tilde{L}(xr) \leq c_1\tilde{L}(r)$ for any $x \in [\frac{1}{2}, 1]$ and $r \geq 1$. This implies also that for any $\rho \in \mathbb{R}$ there exists $c = c(\rho)$ such that $(xr)^\rho \tilde{L}(xr) \leq cr^\rho \tilde{L}(r)$ for any $x \in [\frac{1}{2}, 1]$ and $r \geq 1$. Therefore in our case, whenever $n - m \geq n/2$ we have that $K(n-m) \leq n^\rho \tilde{L}(n)e^{-\mathsf{c}(n-m)^\gamma} = K(n)e^{R(n)-R(n-m)}$. Summarizing, by using all these observations we conclude that

$$\sum_{m=h(n)}^{\frac{n}{2}} \frac{K(n-m)K(m)}{K(n)} \leq c \sum_{m=h(n)}^{\infty} m^\rho \tilde{L}(m)e^{-\mathsf{c}(1-\gamma)R(m)}, \tag{4.119}$$

which goes to 0 as $h(n) \to \infty$ and the proof follows by observing that

$$\sum_{m=h(n)}^{n-h(n)} \frac{K(n-m)K(m)}{K(n)} = 2 \sum_{m=h(n)}^{\frac{n}{2}} \frac{K(n-m)K(m)}{K(n)}. \tag{4.120}$$

□

5 Universality for the pinning model in the weak coupling regime

In this section we prove the results discussed in Chapter 3.2.

We consider disordered pinning models, when the underlying return time distribution has a polynomial tail with exponent $\alpha \in (\frac{1}{2}, 1)$. We show that the free energy and critical curve have an explicit universal asymptotic behavior in the weak coupling regime, depending only on the tail of the return time distribution and not on finer details of the models. This is obtained comparing the partition functions with corresponding continuum quantities, through coarse-graining techniques.

The pre-print [25] has been taken from the content of this chapter .

5.1 Introduction and motivation

Understanding the effect of disorder is a key topic in statistical mechanics, dating back at least to the seminal work of Harris [51]. For models that are disorder relevant, i.e. for which an arbitrary amount of disorder modifies the critical properties, it was recently shown in [23] that it is interesting to look at a suitable *continuum and weak disorder regime*, tuning the disorder strength to zero as the size of the system diverges, which leads to a *continuum model* in which disorder is still present. This framework includes many interesting models, including the 2d random field Ising model with site disorder, the disordered pinning model and the directed polymer in random environment (which was previously considered by Alberts, Quastel and Khanin [2, 1]).

Heuristically, a continuum model should capture the properties of a large family of discrete models, leading to sharp predictions about the scaling behavior of key quantities, such free energy and critical curve, in the weak disorder regime. The goal of this chapter is to make this statement rigorous in the context of disordered pinning models [44, 45, 31], sharpening the available estimates in the literature and proving a form of universality. Although we stick to pinning models, the main ideas have a general value and should be applicable to other models as well.

In this section we give a concise description of our results, focusing on the critical curve. Our complete results are presented in the next section. Throughout the chapter we use the conventions $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and we write $a_n \sim b_n$ to mean $\lim_{n\to\infty} a_n/b_n = 1$.

To build a disordered pinning model, we take a Markov chain $(S = (S_n)_{n\in\mathbb{N}_0}, \mathrm{P})$ starting at a distinguished state, called 0, and we modify its distribution by rewarding/penalizing each visit to 0. The rewards/penalties are determined by a sequence of i.i.d. real random variables $(\omega = (\omega_n)_{n\in\mathbb{N}}, \mathbb{P})$, independent of S, called *disorder variables* (or *charges*). We make the following assumptions.

- The return time to 0 of the Markov chain $\tau_1 := \min\{n \in \mathbb{N} : S_n = 0\}$ satisfies

$$\mathrm{P}(\tau_1 < \infty) = 1, \qquad K(n) := \mathrm{P}(\tau_1 = n) \sim \frac{L(n)}{n^{1+\alpha}}, \quad n \to \infty, \tag{5.1}$$

  where $\alpha \in (0, \infty)$ and $L(n)$ is a slowly varying function [14]. For simplicity we assume that $K(n) > 0$ for all $n \in \mathbb{N}$, but periodicity can be easily dealt with (e.g. $K(n) > 0$ iff $n \in 2\mathbb{N}$).

- The disorder variables have locally finite exponential moments:

$$\exists \beta_0 > 0 : \ \Lambda(\beta) := \log \mathbb{E}(e^{\beta\omega_1}) < \infty, \ \forall \beta \in (-\beta_0, \beta_0), \qquad \mathbb{E}(\omega_1) = 0, \qquad \mathbb{V}(\omega_1) = 1\,, \tag{5.2}$$

  where the choice of zero mean and unit variance is just a convenient normalization.

Given a $\mathbb{P}$-typical realization of the sequence $\omega = (\omega_n)_{n\in\mathbb{N}}$, the *pinning model* is defined as the following random probability law $P^{\omega}_{\beta,h,N}$ on Markov chain paths S:

$$\frac{dP^{\omega}_{\beta,h,N}}{dP}(S) := \frac{e^{\sum_{n=1}^{N}(\beta\omega_n-\Lambda(\beta)+h)\mathbb{1}_{\{S_n=0\}}}}{Z^{\omega}_{\beta,h}(N)}, \qquad Z^{\omega}_{\beta,h}(N) := E\Big[e^{\sum_{n=1}^{N}(\beta\omega_n-\Lambda(\beta)+h)\mathbb{1}_{\{S_n=0\}}}\Big], \tag{5.3}$$

where $N \in \mathbb{N}$ represents the "system size" while $\beta \geq 0$ and $h \in \mathbb{R}$ tune the disorder strength and bias. (The factor $\Lambda(\beta)$ in (5.3) is just a translation of h, introduced so that $\mathbb{E}[e^{\beta\omega_n-\Lambda(\beta)}] = 1$.)

Fixing $\beta \geq 0$ and varying h, the pinning model undergoes a localization/delocalization *phase transition* at a critical value $h_c(\beta) \in \mathbb{R}$: the typical paths S under $P^{\omega}_{\beta,h,N}$ are localized at 0 for $h > h_c(\beta)$, while they are delocalized away from 0 for $h < h_c(\beta)$ (see (5.16) below for a precise result).

It is known that $h_c(\cdot)$ is a continuous function, with $h_c(0) = 0$ (note that for $\beta = 0$ the disorder ω disappears in (5.3) and one is left with a homogeneous model, which is exactly solvable). The behavior of $h_c(\beta)$ as $\beta \to 0$ has been investigated in depth [46, 3, 32, 4, 26], confirming the so-called *Harris criterion* [51]: recalling that α is the tail exponent in (5.1), it was shown that:

- for $\alpha < \frac{1}{2}$ one has $h_c(\beta) \equiv 0$ for $\beta > 0$ small enough (irrelevant disorder regime);
- for $\alpha > \frac{1}{2}$, on the other hand, one has $h_c(\beta) > 0$ for all $\beta > 0$. Moreover, it was proven [49] that disorder changes the order of the phase transition: free energy vanishes for $h \downarrow h_c(\beta)$ at least as fast as $(h - h_c(\beta))^2$, while for $\beta = 0$ the critical exponent is $\max(1/\alpha, 1) < 2$. This case is therefore called *relevant disorder regime*;
- for $\alpha = \frac{1}{2}$, known as the "marginal" case, the answer depends on the slowly varying function $L(\cdot)$ in (5.1): more precisely one has disorder relevance if and only if $\sum_n \frac{1}{n\,(L(n))^2} = \infty$, as recently proved in [10] (see also [4, 46, 47] for previous partial results).

In the special case $\alpha > 1$, when the mean return time $E[\tau_1]$ is finite, one has (cf. [9])

$$\lim_{\beta\to 0} \frac{h_c(\beta)}{\beta^2} = \frac{1}{2E[\tau_1]}\,\frac{\alpha}{1+\alpha}\,. \tag{5.4}$$

In this chapter we focus on the case $\alpha \in (\frac{1}{2}, 1)$, where the mean return time is infinite: $E[\tau_1] = \infty$. In this case, the precise asymptotic behavior of $h_c(\beta)$ as $\beta \to 0$ was known only up to non-matching constants, cf. [3, 32]: there is a slowly varying function $\tilde{L}_\alpha$ (determined explicitly by L and α) and constants $0 < c < C < \infty$ such that for $\beta > 0$ small enough

$$c\,\tilde{L}_\alpha(\tfrac{1}{\beta})\,\beta^{\frac{2\alpha}{2\alpha-1}} \leq h_c(\beta) \leq C\,\tilde{L}_\alpha(\tfrac{1}{\beta})\,\beta^{\frac{2\alpha}{2\alpha-1}}\,. \tag{5.5}$$

Our key result (Theorem 5.4 below) shows that this relation can be made sharp: there exists $m_\alpha \in (0, \infty)$ such that, under mild assumptions on the return time and disorder distributions,

$$\lim_{\beta\to 0} \frac{h_c(\beta)}{\tilde{L}_\alpha(\frac{1}{\beta})\,\beta^{\frac{2\alpha}{2\alpha-1}}} = m_\alpha. \tag{5.6}$$

Let us stress the *universality* value of (5.6): the asymptotic behavior of $h_c(\beta)$ as $\beta \to 0$ depends only on the *tail* of the return time distribution $K(n) = P(\tau_1 = n)$, through the exponent α and the slowly varying function L appearing in (5.1) (which determine $\tilde{L}_\alpha$): all finer details of $K(n)$ beyond these key features disappear in the weak disorder regime. The same holds for the disorder variables: any admissible distribution for ω_1 has the same effect on the asymptotic behavior of $h_c(\beta)$.

Unlike (5.4), we do not know the explicit value of the limiting constant m_α in (5.6), but we can characterize it as the critical parameter of the *continuum disordered pinning model* (CDPM) recently introduced in [24, 23]. The core of our approach is a precise quantitative comparison between discrete pinning models and the CDPM, or more precisely between the corresponding partition functions, based on a subtle *coarse-graining* procedure which extends the one developed in [16, 22] for the copolymer model. This extension turns out to be quite subtle, because unlike the copolymer case *the CDPM admits no "continuum Hamiltonian"*: although it is built over the α-stable regenerative set (which is the continuum limit of renewal processes satisfying (5.1), see §5.5.2), its law is *not* absolutely continuous with respect to the law of the regenerative set, cf. [24]. As a consequence, we need to introduce a suitable coarse-grained Hamiltonian, based on partition functions, which behaves well in the continuum limit. This extension of the coarse-graining procedure is of independent interest and should be applicable to other models with no "continuum Hamiltonian", including the directed polymer in random environment [1].

Overall, our results reinforce the role of the CDPM as a universal model, capturing the key properties of discrete pinning models in the weak coupling regime.

5.2 Main results

5.2.1 Pinning model revisited

The disordered pinning model $\mathrm{P}^{\omega}_{\beta,h,N}$ was defined in (5.3) as a perturbation of a Markov chain S. Since the interaction only takes place when $S_n = 0$, it is customary to forget about the full Markov chain path, focusing only on its zero level set

$$\tau = \{n \in \mathbb{N}_0 : \; S_n = 0\},$$

that we look at as a random subset of $\mathbb{N}_0$. Denoting by $0 = \tau_0 < \tau_1 < \tau_2 < \ldots$ the points of τ, we have a *renewal process* $(\tau_k)_{k\in\mathbb{N}_0}$, i.e. the random variables $(\tau_j - \tau_{j-1})_{j\in\mathbb{N}}$ are i.i.d. with values in $\mathbb{N}$. Note that we have the equality $\{S_n = 0\} = \{n \in \tau\}$, where we use the shorthand

$$\{n \in \tau\} := \bigcup_{k\in\mathbb{N}_0} \{\tau_k = n\}.$$

Consequently, viewing the pinning model $\mathrm{P}^{\omega}_{\beta,h,N}$ as a law for τ, we can rewrite (5.3) as follows:

$$\frac{\mathrm{d}\mathrm{P}^{\omega}_{\beta,h,N}}{\mathrm{d}\mathrm{P}}(\tau) := \frac{e^{\sum_{n=1}^{N}(\beta\omega_n-\Lambda(\beta)+h)\mathbb{1}_{\{n\in\tau\}}}}{\mathrm{Z}^{\omega}_{\beta,h}(N)}, \qquad \mathrm{Z}^{\omega}_{\beta,h}(N) := \mathrm{E}\Big[e^{\sum_{n=1}^{N}(\beta\omega_n-\Lambda(\beta)+h)\mathbb{1}_{\{n\in\tau\}}}\Big]. \tag{5.7}$$

To summarize, henceforth we fix a renewal process $(\tau = (\tau_k)_{k\in\mathbb{N}_0}, \mathrm{P})$ satisfying (5.1) and an i.i.d. sequence of disorder variables $(\omega = (\omega_n)_{n\in\mathbb{N}}, \mathbb{P})$ satisfying (5.2). We then define the disordered pinning model as the random probability law $\mathrm{P}^{\omega}_{\beta,h,N}$ for τ defined in (5.7).

In order to prove our results, we need some additional assumptions. We recall that for any renewal process satisfying (5.1) with $\alpha \in (0, 1)$, the following local renewal theorem holds [42, 34]:

$$u(n) := \mathrm{P}(n \in \tau) \sim \frac{C_\alpha}{L(n)\, n^{1-\alpha}}, \quad n \to \infty, \qquad \text{with } \; C_\alpha := \frac{\alpha \sin(\alpha\pi)}{\pi}. \tag{5.8}$$

In particular, if $\ell = o(n)$, then $u(n + \ell)/u(n) \to 1$ as $n \to \infty$. We are going to assume that this convergence takes place at a not too slow rate, i.e. at least a power law of $\frac{\ell}{n}$, as in [24, eq. (1.7)]:

$$\exists\, C, n_0 \in (0, \infty);\; \varepsilon, \delta \in (0, 1] : \quad \left|\frac{u(n+\ell)}{u(n)} - 1\right| \le C\left(\frac{\ell}{n}\right)^{\delta}, \quad \forall n \ge n_0,\ 0 \le \ell \le \varepsilon n. \tag{5.9}$$

Remark 5.1. This is a mild assumption, as discussed in [24, Appendix B]. For instance, one can build a wide family of nearest-neighbor Markov chains on $\mathbb{N}_0$ with ±1 increments (Bessel-like random walks) satisfying (5.1), cf. [5], and in this case (5.9) holds for any $\delta < \alpha$.

Concerning the disorder distribution, we strengthen the finite exponential moment assumption

(5.2), requiring the following concentration inequality:

$$\exists \gamma > 1, C_1, C_2 \in (0,\infty): \text{ for all } n \in \mathbb{N} \text{ and for all } f : \mathbb{R}^n \to \mathbb{R} \text{ convex and 1-Lipschitz}$$
$$\mathbb{P}\Big(\big|f(\omega_1,\ldots,\omega_n) - M_f\big| \geq t\Big) \leq C_1 \exp\Big(-\frac{t^\gamma}{C_2}\Big), \tag{5.10}$$

where 1-Lipschitz means $|f(x) - f(y)| \leq |x - y|$ for all $x, y \in \mathbb{R}^n$, with $|\cdot|$ the usual Euclidean norm, and M_f denotes a median of $f(\omega_1,\ldots,\omega_n)$. (One can equivalently take M_f to be the mean $\mathbb{E}[f(\omega_1,\ldots,\omega_n)]$ just by changing the constants C_1, C_2, cf. [63, Proposition 1.8].)

It is known that (5.10) holds under fairly general assumptions, namely:

- $(\gamma = 2)$ if ω_1 is bounded, i.e. $\mathbb{P}(|\omega_1| \leq a) = 1$ for some $a \in (0,\infty)$, cf. [63, Corollary 4.10];
- $(\gamma = 2)$ if the law of ω_1 satisfies a log-Sobolev inequality, in particular if ω_1 is Gaussian, cf. [63, Theorems 5.3 and Corollary 5.7]; more generally, if the law of ω_1 is absolutely continuous with density $\exp(-U - V)$, where U is uniformly strictly convex (i.e. $U(x) - cx^2$ is convex, for some $c > 0$) and V is bounded, cf. [63, Theorems 5.2 and Proposition 5.5];
- $(\gamma \in (1,2))$ if the law of ω_1 is absolutely continuous with density given by $c_\gamma e^{-|x|^\gamma}$ (see Propositions 4.18 and 4.19 in [63] and the following considerations).

5.2.2 Free energy and critical curve

The normalization constant $Z^\omega_{\beta,h}(N)$ in (5.7) is called *partition function* and plays a key role. Its rate of exponential growth as $N \to \infty$ is called *free energy*:

$$\mathrm{F}(\beta,h) := \lim_{N\to\infty} \frac{1}{N} \log Z^\omega_{\beta,h}(N) = \lim_{N\to\infty} \frac{1}{N} \mathbb{E}\Big[\log Z^\omega_{\beta,h}(N)\Big], \qquad \mathbb{P}\text{-a.s. and in } L^1, \tag{5.11}$$

where the limit exists and is finite by super-additive arguments [44, 31]. Let us stress that $\mathrm{F}(\beta,h)$ depends on the laws of the renewal process $\mathrm{P}(\tau_1 = n)$ and of the disorder variables $\mathbb{P}(\omega_1 \in \mathrm{d}x)$, but it does not depend on the $\mathbb{P}$-typical realization of the sequence $(\omega_n)_{n\in\mathbb{N}}$. Also note that $h \mapsto \mathrm{F}(\beta,h)$ inherits from $h \mapsto \log Z^\omega_{\beta,h}(N)$ the properties of being convex and non-decreasing.

Restricting the expectation defining $Z^\omega_{\beta,h}(N)$ to the event $\{\tau_1 > N\}$ and recalling the polynomial tail assumption (5.1), one obtains the basic but crucial inequality

$$\mathrm{F}(\beta,h) \geq 0 \quad \forall \beta \geq 0, h \in \mathbb{R}. \tag{5.12}$$

One then defines the *critical curve* by

$$h_c(\beta) := \sup\{h \in \mathbb{R} : \ \mathrm{F}(\beta,h) = 0\}. \tag{5.13}$$

It can be shown that $0 < h_c(\beta) < \infty$ for $\beta > 0$, and by monotonicity and continuity in h one has

$$\mathrm{F}(\beta,h) = 0 \ \text{ if } \ h \leq h_c(\beta), \qquad \mathrm{F}(\beta,h) > 0 \ \text{ if } \ h > h_c(\beta). \tag{5.14}$$

In particular, the function $h \mapsto \mathrm{F}(\beta,h)$ is non-analytic at the point $h_c(\beta)$, which is called a *phase transition* point. A probabilistic interpretation can be given looking at the quantity

$$\ell_N := \sum_{n=1}^N \mathbb{1}_{\{n\in\tau\}} = \big|\tau \cap (0,N]\big|, \tag{5.15}$$

which represents the number of points of $\tau \cap (0,N]$. By convexity, $h \mapsto \mathrm{F}(\beta,h)$ is differentiable at all but a countable number of points, and for pinning models it can be shown that it is actually C^∞ for

$h \neq h_c(\beta)$ [48]. Interchanging differentiation and limit in (5.11), by convexity, relation (5.7) yields

$$\text{for } \mathbb{P}\text{-a.e. } \omega, \qquad \lim_{N\to\infty} \mathrm{E}^{\omega}_{\beta,h,N}\left[\frac{\ell_N}{N}\right] = \frac{\partial \mathrm{F}(\beta,h)}{\partial h} \begin{cases} = 0 & \text{if } h < h_c(\beta) \\ > 0 & \text{if } h > h_c(\beta) \end{cases}. \tag{5.16}$$

This shows that the typical paths of the pinning model are indeed localized at 0 for $h > h_c(\beta)$ and delocalized away from 0 for $h < h_c(\beta)$.* We refer to [44, 45, 31] for details and for finer results.

5.2.3 Main results

Our goal is to study the asymptotic behavior of the free energy $\mathrm{F}(\beta, h)$ and critical curve $h_c(\beta)$ in the weak coupling regime $\beta, h \to 0$.

Let us recall the recent results in [23, 24], which are the starting point of our analysis. Consider any disordered pinning model where the renewal process satisfies (5.1), with $\alpha \in (\frac{1}{2}, 1)$, and the disorder satisfies (5.2). If we let $N \to \infty$ and simultaneously $\beta \to 0$, $h \to 0$ as follows:

$$\beta = \beta_N := \hat{\beta}\frac{L(N)}{N^{\alpha-\frac{1}{2}}}, \qquad h = h_N := \hat{h}\frac{L(N)}{N^{\alpha}}, \qquad \text{for fixed } \hat{\beta} > 0,\ \hat{h} \in \mathbb{R}, \tag{5.17}$$

the family of partition functions $\mathrm{Z}^{\omega}_{\beta_N,h_N}(Nt)$, with $t \in [0, \infty)$, has a universal limit, in the sense of finite-dimensional distributions [23, Theorem 3.1]:

$$\left(\mathrm{Z}^{\omega}_{\beta_N,h_N}(Nt)\right)_{t\in[0,\infty)} \xrightarrow{(\mathrm{d})} \left(\mathbf{Z}^{W}_{\hat{\beta},\hat{h}}(t)\right)_{t\in[0,\infty)}, \qquad N \to \infty. \tag{5.18}$$

The *continuum partition function* $\mathbf{Z}^{W}_{\hat{\beta},\hat{h}}(t)$ depends only on the exponent α and on a Brownian motion $(W = (W_t)_{t\geq 0}, \mathbb{P})$, playing the role of continuum disorder. We point out that $\mathbf{Z}^{W}_{\hat{\beta},\hat{h}}(t)$ has an explicit Wiener chaos representation, as a series of deterministic and stochastic integrals (see (5.63) below), and admits a version which is continuous in t, that we fix henceforth (see §5.2.5 for more details).

Remark 5.2. For an intuitive explanation of why β_N, h_N should scale as in (5.17), we refer to the discussion following Theorem 1.3 in [24]. Alternatively, one can invert the relations in (5.17), for simplicity in the case $\hat{\beta} = 1$, expressing N and h as a function of β as follows:

$$\frac{1}{N} \sim \tilde{L}_{\alpha}(\tfrac{1}{\beta})^2 \beta^{\frac{2}{2\alpha-1}}, \qquad h \sim \hat{h}\,\tilde{L}_{\alpha}(\tfrac{1}{\beta})\beta^{\frac{2\alpha}{2\alpha-1}}, \tag{5.19}$$

where $\tilde{L}_{\alpha}$ is the same slowly varying function appearing in (5.5), determined explicitly by L and α. Thus $h = h_N$ is of the same order as the critical curve $h_c(\beta_N)$, which is quite a natural choice.

More precisely, one has $\tilde{L}_{\alpha}(x) = M^{\#}(x)^{-\frac{1}{2\alpha-1}}$, where $M^{\#}$ is the *de Bruijn conjugate* of the slowly varying function $M(x) := 1/L(x^{\frac{2}{2\alpha-1}})$, cf. [14, Theorem 1.5.13], defined by the asymptotic property $M^{\#}(xM(x)) \sim 1/M(x)$. We refer to (3.17) in [23] and the following lines for more details.

It is natural to define a *continuum free energy* $\mathbf{F}^{\alpha}(\hat{\beta}, \hat{h})$ in terms of $\mathbf{Z}^{W}_{\hat{\beta},\hat{h}}(t)$, in analogy with (5.11). Our first result ensures the existence of such a quantity along $t \in \mathbb{N}$, if we average over the disorder. One can also show the existence of such limit, without restrictions on t, in the $\mathbb{P}(\mathrm{d}W)$-a.s. and L^1 senses: we refer to Chapter 6 for a proof.

Theorem 5.3 (Continuum free energy). *For all $\alpha \in (\frac{1}{2}, 1)$, $\hat{\beta} > 0$, $\hat{h} \in \mathbb{R}$ the following limit exists and is finite:*

$$\mathbf{F}^{\alpha}(\hat{\beta}, \hat{h}) := \lim_{t\to\infty,\, t\in\mathbb{N}} \frac{1}{t}\, \mathbb{E}\left[\log \mathbf{Z}^{W}_{\hat{\beta},\hat{h}}(t)\right]. \tag{5.20}$$

*Note that, in Markov chain terms, ℓ_N is the number of visits of S to the state 0, up to time N.

The function $\mathbf{F}^\alpha(\hat\beta,\hat h)$ *is non-negative:* $\mathbf{F}^\alpha(\hat\beta,\hat h) \ge 0$ *for all* $\hat\beta > 0$, $\hat h \in \mathbb{R}$. *Furthermore, it is a convex function of* $\hat h$, *for fixed* $\hat\beta$, *and satisfies the following scaling relation:*

$$\mathbf{F}^\alpha(c^{\alpha-\frac{1}{2}}\hat\beta, c^\alpha \hat h) = c\,\mathbf{F}^\alpha(\hat\beta,\hat h)\,, \qquad \forall \hat\beta > 0,\ \hat h \in \mathbb{R},\ c \in (0,\infty)\,. \tag{5.21}$$

In analogy with (5.13), we define the *continuum critical curve* $\mathbf{h}_c^\alpha(\hat\beta)$ by

$$\mathbf{h}_c^\alpha(\hat\beta) = \sup\{\hat h \in \mathbb{R} : \ \mathbf{F}^\alpha(\hat\beta,\hat h) = 0\}, \tag{5.22}$$

which turns out to be positive and finite (see Remark 5.5 below). Note that, by (5.21),

$$\mathbf{F}^\alpha(\hat\beta,\hat h) = \mathbf{F}^\alpha\Big(1, \frac{\hat h}{\hat\beta^{\frac{2\alpha}{2\alpha-1}}}\Big)\hat\beta^{\frac{2}{2\alpha-1}}\,, \qquad \text{hence} \qquad \mathbf{h}_c^\alpha(\hat\beta) = \mathbf{h}_c^\alpha(1)\,\hat\beta^{\frac{2\alpha}{2\alpha-1}}\,. \tag{5.23}$$

Heuristically, the continuum free energy $\mathbf{F}^\alpha(\hat\beta,\hat h)$ and critical curve $\mathbf{h}_c^\alpha(\hat\beta)$ capture the asymptotic behavior of their discrete counterparts $\mathrm{F}(\beta,h)$ and $h_c(\beta)$ in the weak coupling regime $h,\beta \to 0$. In fact, the convergence in distribution (5.18) suggests that

$$\mathbb{E}\Big[\log \mathbf{Z}^W_{\hat\beta,\hat h}(t)\Big] = \lim_{N\to\infty} \mathbb{E}\Big[\log \mathrm{Z}^\omega_{\beta_N,h_N}(Nt)\Big]. \tag{5.24}$$

Plugging (5.24) into (5.20) and *interchanging the limits* $t \to \infty$ *and* $N \to \infty$ would yield

$$\mathbf{F}^\alpha(\hat\beta,\hat h) = \lim_{t\to\infty}\frac{1}{t}\lim_{N\to\infty}\mathbb{E}\Big[\log \mathrm{Z}^\omega_{\beta_N,h_N}(Nt)\Big] = \lim_{N\to\infty} N \lim_{t\to\infty}\frac{1}{Nt}\mathbb{E}\Big[\log \mathrm{Z}^\omega_{\beta_N,h_N}(Nt)\Big], \tag{5.25}$$

which by (5.11) and (5.17) leads to the key relation (with $\varepsilon = \frac{1}{N}$):

$$\mathbf{F}^\alpha(\hat\beta,\hat h) = \lim_{N\to\infty} N\,\mathrm{F}(\beta_N,h_N) = \lim_{\varepsilon\downarrow 0}\frac{\mathrm{F}(\hat\beta\,\varepsilon^{\alpha-\frac{1}{2}}L(\frac{1}{\varepsilon}),\ \hat h\,\varepsilon^\alpha L(\frac{1}{\varepsilon}))}{\varepsilon}. \tag{5.26}$$

We point out that relation (5.24) is typically justified, as the family $(\log \mathrm{Z}^\omega_{\beta_N,h_N}(Nt))_{N\in\mathbb{N}}$ can be shown to be uniformly integrable, but the interchanging of limits in (5.25) is in general a delicate issue. This was shown to hold for the copolymer model with tail exponent $\alpha < 1$, cf. [16, 22], but it is known to *fail* for both pinning and copolymer models with $\alpha > 1$ (see point 3 in [23, §1.3]).

The following theorem, which is our main result, shows that for disordered pinning models with $\alpha \in (\frac{1}{2},1)$ relation (5.26) does hold. We actually prove a stronger relation, which also yields the precise asymptotic behavior of the critical curve.

Theorem 5.4 (Interchanging the limits). *Let* $\mathrm{F}(\beta,h)$ *be the free energy of the disordered pinning model* (5.7)-(5.11)*, where the renewal process* τ *satisfies* (5.1)-(5.9) *for some* $\alpha \in (\frac{1}{2},1)$ *and the disorder* ω *satisfies* (5.2)-(5.10). *For all* $\hat\beta > 0$, $\hat h \in \mathbb{R}$ *and* $\eta > 0$ *there exists* $\varepsilon_0 > 0$ *such that*

$$\mathbf{F}^\alpha\left(\hat\beta,\hat h-\eta\right) \le \frac{\mathrm{F}(\hat\beta\,\varepsilon^{\alpha-\frac{1}{2}}L(\frac{1}{\varepsilon}),\ \hat h\,\varepsilon^\alpha L(\frac{1}{\varepsilon}))}{\varepsilon} \le \mathbf{F}^\alpha\left(\hat\beta,\hat h+\eta\right), \qquad \forall \varepsilon \in (0,\varepsilon_0)\,. \tag{5.27}$$

As a consequence, relation (5.26) *holds, and furthermore*

$$\lim_{\beta\to 0}\frac{h_c(\beta)}{\tilde L_\alpha(\frac{1}{\beta})\beta^{\frac{2\alpha}{2\alpha-1}}} = \mathbf{h}_c^\alpha(1), \tag{5.28}$$

where $\tilde L_\alpha$ *is the slowly function appearing in* (5.19) *and the following lines.*

Note that relation (5.26) follows immediately by (5.27), sending first $\varepsilon \to 0$ and then $\eta \to 0$, because $\hat h \mapsto \mathbf{F}^\alpha(\hat\beta,\hat h)$ is continuous (by convexity, cf. Theorem 5.3). Relation (5.28) also follows by

(5.27), cf. §5.5.1, but it would not follow from (5.26), because convergence of functions does not necessarily imply convergence of the respective zero level sets. This is why we prove (5.27).

Remark 5.5. Relation (5.28), coupled with the known bounds (5.5) from the literature, shows in particular that $0 < \mathbf{h}_c^\alpha(1) < \infty$ (hence $0 < \mathbf{h}_c^\alpha(\hat\beta) < \infty$ for every $\hat\beta > 0$, by (5.23)). Of course, in principle this can be proved by direct estimates on the continuum partition function.

5.2.4 On the critical behavior

Fix $\hat\beta > 0$. The scaling relations (5.23) imply that for all $\varepsilon > 0$

$$\mathbf{F}^\alpha(\hat\beta, \mathbf{h}_c^\alpha(\hat\beta) + \varepsilon) = \hat\beta^{\frac{2}{2\alpha-1}}\, \mathbf{F}^\alpha\left(1, \mathbf{h}_c^\alpha(1) + \frac{\varepsilon}{\hat\beta^{\frac{2\alpha}{2\alpha-1}}}\right).$$

Thus, as $\varepsilon \downarrow 0$ (i.e. as $\hat h \downarrow \mathbf{h}_c^\alpha(\hat\beta)$) the free energy vanishes in the same way; in particular, *the critical exponent γ is the same for every $\hat\beta$* (provided it exists):

$$\mathbf{F}^\alpha(1, \hat h) \underset{\hat h \downarrow \mathbf{h}_c^\alpha(1)}{=} (\hat h - \mathbf{h}_c^\alpha(1))^{\gamma + o(1)} \qquad \Longrightarrow \qquad \mathbf{F}^\alpha(\hat\beta, \hat h) \underset{\hat h \downarrow \mathbf{h}_c^\alpha(1)}{=} \hat\beta^{\frac{-2(\alpha\gamma-1)}{2\alpha-1}} (\hat h - \mathbf{h}_c^\alpha(\hat\beta))^{\gamma+o(1)}. \tag{5.29}$$

Another interesting observation is that the smoothing inequality of [49] can be extended to the continuum. For instance, in the case of Gaussian disorder $\omega_i \sim N(0,1)$, it is known that the discrete free energy $\mathrm{F}(\beta, h)$ satisfies the following relation, for all $\beta > 0$ and $h \in \mathbb{R}$:

$$0 \le \mathrm{F}(\beta, h) \le \frac{1+\alpha}{2\beta^2} (h - h_c(\beta))^2.$$

Consider a renewal process satisfying (5.1) with $L \equiv 1$ (so that also $\tilde L_\alpha \equiv 1$, cf. Remark 5.2). Choosing $\beta = \hat\beta\, \varepsilon^{\alpha - \frac12}$ and $h = \hat h\, \varepsilon^\alpha$ and letting $\varepsilon \downarrow 0$, we can apply our key results (5.26) and (5.28) (recall also (5.23)), obtaining a smoothing inequality for the continuum free energy:

$$\mathbf{F}^\alpha(\hat\beta, \hat h) \le \frac{1+\alpha}{2\hat\beta^2} \left(\hat h - \mathbf{h}_c^\alpha(\hat\beta)\right)^2.$$

In particular, the exponent γ in (5.29) has to satisfy $\gamma \ge 2$ (and consequently, the prefactor in the second relation in (5.29) is $\hat\beta^{-\eta}$ with $\eta > 0$).

5.2.5 Further results

Our results on the free energy and critical curve are based on a comparison of discrete and continuum partition function, whose properties we investigate in depth. Some of the results of independent interest are presented here.

Alongside the "free" partition function $\mathrm{Z}_{\beta,h}^\omega(N)$ in (5.7), it is useful to consider a family $\mathrm{Z}_{\beta,h}^{\omega,\mathtt{c}}(a,b)$ of "conditioned" partition functions, for $a, b \in \mathbb{N}_0$ with $a \le b$:

$$\mathrm{Z}_{\beta,h}^{\omega,\mathtt{c}}(a,b) = \mathrm{E}\left(e^{\sum_{k=a+1}^{b-1}(\beta\omega_k - \Lambda(\beta) + h)\mathbb{1}_{k\in\tau}} \,\middle|\, a \in \tau, b \in \tau\right). \tag{5.30}$$

If we let $N \to \infty$ with β_N, h_N as in (5.17), the partition functions $\mathrm{Z}_{\beta_N,h_N}^{\omega,\mathtt{c}}(Ns, Nt)$, for (s,t) in

$$[0,\infty)^2_\le := \{(s,t) \in [0,\infty)^2 \mid s \le t\},$$

converge in the sense of finite-dimensional distributions [23, Theorem 3.1], in analogy with (5.18):

$$\left(\mathrm{Z}_{\beta_N,h_N}^{\omega,\mathtt{c}}(Ns,Nt)\right)_{(s,t)\in[0,\infty)^2_\le} \xrightarrow{(\mathrm{d})} \left(\mathbf{Z}_{\hat\beta,\hat h}^{W,\mathtt{c}}(s,t)\right)_{(s,t)\in[0,\infty)^2_\le}, \qquad N \to \infty, \tag{5.31}$$

where $\mathbf{Z}^{W,\mathrm{c}}_{\hat\beta,\hat h}(s,t)$ admits an explicit Wiener chaos expansion, cf. (5.64) below.

It was shown in [24, Theorem 2.1 and Remark 2.3] that, under the further assumption (5.9), the convergences (5.18) and (5.31) can be upgraded: by linearly interpolating the discrete partition functions for $Ns, Nt \notin \mathbb{N}_0$, one has convergence in distribution in the space of continuous functions of $t \in [0,\infty)$ and of $(s,t) \in [0,\infty)^2_{\le}$, respectively, equipped with the topology of uniform convergence on compact sets. We strengthen this result, by showing that the convergence is locally uniform also in the variable $\hat h \in \mathbb{R}$. We formulate this fact through the existence of a suitable coupling.

Theorem 5.6 (Uniformity in $\hat h$)**.** *Assume* (5.1)-(5.9)*, for some* $\alpha \in (\frac{1}{2},1)$*, and* (5.2)*. For all* $\hat\beta > 0$*, there is a coupling of discrete and continuum partition functions such that the convergence* (5.18)*, resp.* (5.31)*, holds* $\mathbb{P}(\mathrm{d}\omega, \mathrm{d}W)$*-a.s. uniformly in any compact set of values of* $(t,\hat h)$*, resp. of* $(s,t,\hat h)$*.*

We prove Theorem 5.6 by showing that partition functions with $\hat h \neq 0$ can be expressed in terms of those with $\hat h = 0$ through an explicit series expansion (see Theorem 5.16 below). This representation shows that the continuum partition functions are increasing in $\hat h$. They are also log-convex in $\hat h$, because $h \mapsto \log \mathrm{Z}^{\omega}_{\beta,h}$ and $h \mapsto \log \mathrm{Z}^{\omega,\mathrm{c}}_{\beta,h}$ are convex functions (by Hölder's inequality, cf. (5.7) and (5.30)) and convexity is preserved by pointwise limits. Summarizing:

Proposition 5.7. *For all* $\alpha \in (\frac{1}{2},1)$ *and* $\hat\beta > 0$*, the process* $\mathbf{Z}^{W}_{\hat\beta,\hat h}(t)$*, resp.* $\mathbf{Z}^{W,\mathrm{c}}_{\hat\beta,\hat h}(s,t)$*, admits a version which is continuous in* $(t,\hat h)$*, resp. in* $(s,t,\hat h)$*. For fixed* $t > 0$*, resp.* $t > s$*, the function* $\hat h \mapsto \log \mathbf{Z}^{W}_{\hat\beta,\hat h}(t)$*, resp.* $\hat h \mapsto \log \mathbf{Z}^{W,\mathrm{c}}_{\hat\beta,\hat h}(s,t)$*, is strictly convex and strictly increasing.*

We conclude with some important estimates, bounding (positive and negative) moments of the partition functions and providing a deviation inequality.

Proposition 5.8. *Assume* (5.1)-(5.9)*, for some* $\alpha \in (\frac{1}{2},1)$*, and* (5.2)*. Fix* $\hat\beta > 0$*,* $\hat h \in \mathbb{R}$*. For all* $T > 0$ *and* $p \in [0,\infty)$*, there exists a constant* $C_{p,T} < \infty$ *such that*

$$\mathbb{E}\left[\sup_{0\le s\le t\le T} \mathrm{Z}^{\omega,\mathrm{c}}_{\beta_N,h_N}(Ns,Nt)^p\right] \le C_{p,T}\,, \qquad \forall N \in \mathbb{N}\,. \tag{5.32}$$

Assuming also (5.10)*, relation* (5.32) *holds also for every* $p \in (-\infty,0]$*, and furthermore one has*

$$\sup_{0\le s\le t\le T} \mathbb{P}\Big(\log \mathrm{Z}^{\omega,\mathrm{c}}_{\beta_N,h_N}(Ns,Nt) \le -x\Big) \le A_T \exp\left(-\frac{x^\gamma}{B_T}\right), \qquad \forall x \ge 0,\ \forall N \in \mathbb{N}\,, \tag{5.33}$$

for suitable finite constants A_T*,* B_T*. Finally, relations* (5.32)*,* (5.33) *hold also for the free partition function* $\mathrm{Z}^{\omega}_{\beta_N,h_N}(Nt)$ *(replacing* $\sup_{0\le s\le t\le T}$ *with* $\sup_{0\le t\le T}$*).*

For relation (5.33) we use the concentration assumptions (5.10) on the disorder. However, since $\log \mathrm{Z}^{\omega,\mathrm{c}}_{\beta_N,h_N}$ is not a uniformly (over $N \in \mathbb{N}$) Lipschitz function of ω, some work is needed.

Finally, since the convergences in distribution (5.18), (5.31) hold in the space of continuous functions, we can easily deduce analogues of (5.32), (5.33) for the continuum partition functions.

Corollary 5.9. *Fix* $\alpha \in (\frac{1}{2},1)$*,* $\hat\beta > 0$*,* $\hat h \in \mathbb{R}$*. For all* $T > 0$ *and* $p \in \mathbb{R}$ *there exist finite constants* A_T*,* B_T*,* $C_{p,T}$ *(depending also on* $\alpha, \hat\beta, \hat h$*) such that*

$$\mathbb{E}\left[\sup_{0\le s\le t\le T} \mathbf{Z}^{W,\mathrm{c}}_{\hat\beta,\hat h}(Ns,Nt)^p\right] \le C_{p,T}\,, \tag{5.34}$$

$$\sup_{0\le s\le t\le T} \mathbb{P}\Big(\log \mathbf{Z}^{W,\mathrm{c}}_{\hat\beta,\hat h}(Ns,Nt) \le -x\Big) \le A_T \exp\left(-\frac{x^\gamma}{B_T}\right), \qquad \forall x \ge 0\,. \tag{5.35}$$

The same relations hold for the free partition function $\mathbf{Z}^{W}_{\hat\beta,\hat h}(t)$ *(replacing* $\sup_{0\le s\le t\le T}$ *with* $\sup_{0\le t\le T}$*).*

5.2.6 Organization of the chapter

The chapter is structured as follows.

- We first prove Proposition 5.8 and Corollary 5.9 in Section 5.3.
- Then we prove Theorem 5.6 in Section 5.4.
- In Section 5.5 we prove our main result, Theorem 5.4. Our approach yields as a by-product the existence of the continuum free energy, i.e. the core of Theorem 5.3.
- The proof of Theorem 5.3 is easily completed in Section 5.6.
- Finally some more technical points have been deferred to the Appendices 5.A and 5.B.

5.3 Proof of Proposition 5.8 and Corollary 5.9

In this section we prove Proposition 5.8. Taking inspiration from [39], we first prove (5.33), using concentration results, and later we prove (5.32). We start with some preliminary results.

5.3.1 Renewal results

Let $(\sigma = (\sigma_n)_{n\in\mathbb{N}_0}, \mathrm{P})$ be a renewal process such that $\mathrm{P}(\sigma_1 = 1) > 0$ and

$$w(n) := \mathrm{P}(n \in \sigma) \overset{n\to\infty}{\sim} \frac{1}{M(n)\, n^{1-\nu}}, \quad \text{with} \quad \nu \in (0,1) \quad \text{and} \quad M(\cdot) \quad \text{slowly varying}. \tag{5.36}$$

This includes any renewal process τ satisfying (5.1) with $\alpha \in (0,1)$, in which case (5.36) holds with $\nu = \alpha$ and $M(n) = L(n)/C_\alpha$, by (5.8). When $\alpha \in (\frac{1}{2}, 1)$, another important example is given by the *intersection renewal* $\sigma = \tau \cap \tau'$, where τ' is an independent copy of τ: since $w(n) = \mathrm{P}(n \in \tau \cap \tau') = \mathrm{P}(n \in \tau)^2$ in this case, by (5.8) relation (5.36) holds with $\nu = 2\alpha - 1$ and $M(n) = L(n)^2/C_\alpha^2$.

For $N \in \mathbb{N}_0$ and $\delta \in \mathbb{R}$, let $\Psi_\delta(N), \Psi_\delta^{\mathtt{c}}(N)$ denote the (deterministic) functions

$$\Psi_\delta(N) = \mathrm{E}\left[e^{\delta \sum_{n=1}^{N} \mathbb{1}_{n\in\sigma}}\right], \qquad \Psi_\delta^{\mathtt{c}}(N) = \mathrm{E}\left[e^{\delta \sum_{n=1}^{N-1} \mathbb{1}_{n\in\sigma}} \,\middle|\, N \in \sigma\right], \tag{5.37}$$

which are just the partition functions of a homogeneous (i.e. non disordered) pinning model. In the next result, which is essentially a deterministic version of [24, Theorem 2.1] (see also [76]), we determine their limits when $N \to \infty$ and $\delta = \delta_N \to 0$ as follows (for fixed $\hat{\delta} \in \mathbb{R}$):

$$\delta_N \sim \hat{\delta}\frac{M(N)}{N^\nu}. \tag{5.38}$$

Theorem 5.10. *Let the renewal σ satisfy* (5.36). *Then the functions* $(\Psi_{\delta_N}(Nt))_{t\in[0,\infty)}$, $(\Psi^{\mathtt{c}}_{\delta_N}(Nt))_{t\in[0,\infty)}$, *with δ_N as in* (5.38) *and linearly interpolated for $Nt \notin \mathbb{N}_0$, converges as $N \to \infty$ respectively to*

$$\mathbf{\Psi}^\nu_{\hat{\delta}}(t) = 1 + \sum_{k=1}^{\infty} \hat{\delta}^k \int\!\!\cdots\!\!\int_{0<t_1<\cdots<t_k<t} \frac{1}{t_1^{1-\nu}(t_2-t_1)^{1-\nu}\cdots(t_k-t_{k-1})^{1-\nu}} \prod_{i=1}^{k} \mathrm{d}t_i\,, \tag{5.39}$$

$$\mathbf{\Psi}^{\nu,\mathtt{c}}_{\hat{\delta}}(t) = 1 + \sum_{k=1}^{\infty} \hat{\delta}^k \int\!\!\cdots\!\!\int_{0<t_1<\cdots<t_k<t} \frac{t^{1-\nu}}{t_1^{1-\nu}(t_2-t_1)^{1-\nu}\cdots(t_k-t_{k-1})^{1-\nu}(t-t_k)^{1-\nu}} \prod_{i=1}^{k} \mathrm{d}t_i\,, \tag{5.40}$$

where the convergence is uniform on compact subsets of $[0,\infty)$. *The limiting functions* $\mathbf{\Psi}^\nu_{\hat{\delta}}(t), \mathbf{\Psi}^{\nu,\mathtt{c}}_{\hat{\delta}}(t)$ *are strictly positive, finite and continuous in t.*

Before proving of Theorem 5.10, we summarize some useful consequences in the next Lemma.

Lemma 5.11. *Let τ be a renewal process satisfying* (5.1) *with $\alpha \in (\frac{1}{2}, 1)$ and let ω satisfy* (5.2). *For every $\hat{\beta} > 0$, $\hat{h} \in \mathbb{R}$, defining β_N, h_N as in* (5.17)*, one has:*

$$\lim_{N\to\infty} \mathbb{E}\left[Z^{\omega,c}_{\beta_N,h_N}(0,Nt)\right] = \mathbf{\Psi}^{\alpha,c}_{C_\alpha \hat{h}}(t)\,, \qquad \lim_{N\to\infty} \mathbb{E}\left[\left(Z^{\omega,c}_{\beta_N,0}(0,Nt)\right)^2\right] = \mathbf{\Psi}^{2\alpha-1,c}_{C_\alpha^2\hat{\beta}^2}(t)\,, \tag{5.41}$$

uniformly on compact subsets of $t \in [0,\infty)$. Consequently

$$\rho := \inf_{N\in\mathbb{N}} \inf_{t\in[0,1]} \mathbb{E}\left[Z^{\omega,c}_{\beta_N,h_N}(0,Nt)\right] > 0\,, \qquad \lambda := \sup_{N\in\mathbb{N}} \sup_{t\in[0,1]} \mathbb{E}\left[\left(Z^{\omega,c}_{\beta_N,0}(0,Nt)\right)^2\right] < \infty\,. \tag{5.42}$$

Analogous results hold for the free partition function.

Proof. We focus on the constrained partition function (the free one is analogous), starting with the first relation in (5.41). By (5.30), for $Nt \in \mathbb{N}_0$ we can write

$$\mathbb{E}\left[Z^{\omega,c}_{\beta_N,h_N}(0,Nt)\right] = \mathrm{E}\left[e^{h_N \sum_{k=1}^{Nt} \mathbb{1}_{k\in\tau}} \,\middle|\, Nt \in \tau\right] = \Psi^{c}_{h_N}(Nt)\,,$$

where we used (5.37) with $\sigma = \tau$. As we observed after (5.36), we have $M(n) = L(n)/C_\alpha$ in this case, so comparing (5.38) with (5.17) we see that $h_N \sim \delta_N$ with $\hat{\delta} = C_\alpha \hat{h}$. Theorem 5.10 then yields (5.41).

Next we prove the second relation in (5.41). Denoting by τ' an independent copy of τ, note that $\mathbb{E}[e^{(\beta\omega_k - \Lambda(\beta))(\mathbb{1}_{k\in\tau}+\mathbb{1}_{k\in\tau'})}] = e^{(\Lambda(2\beta)-2\Lambda(\beta))\mathbb{1}_{k\in\tau\cap\tau'}}$. Then, again by (5.30), for $h_N = 0$ we can write

$$\begin{aligned} \mathbb{E}\left[\left(Z^{\omega,c}_{\beta_N,0}(0,Nt)\right)^2\right] &= \mathbb{E}\left[\mathrm{E}\left[e^{\sum_{k=1}^{Nt-1}(\beta_N\omega_k - \Lambda(\beta_N))(\mathbb{1}_{k\in\tau}+\mathbb{1}_{k\in\tau'})} \,\middle|\, Nt \in \tau\cap\tau'\right]\right] \\ &= \mathrm{E}\left[e^{(\Lambda(2\beta_N)-2\Lambda(\beta_N))\sum_{k=1}^{Nt} \mathbb{1}_{k\in\tau\cap\tau'}} \,\middle|\, Nt \in \tau\cap\tau'\right] = \Psi^{c}_{\Lambda(2\beta_N)-2\Lambda(\beta_N)}(Nt)\,, \end{aligned} \tag{5.43}$$

where in the last equality we have applied (5.37) with $\sigma = \tau\cap\tau'$, for which $\nu = 2\alpha - 1$ and $M(n) = L(n)^2/C_\alpha^2$. Since $\Lambda(\beta) = \frac{1}{2}\beta^2 + o(\beta^2)$ as $\beta \to 0$, by (5.2), it follows that $\Lambda(2\beta_N) - 2\Lambda(\beta_N) \sim \beta_N^2 \sim \delta_N$ with $\hat{\delta} = C_\alpha^2\hat{\beta}^2$, by (5.17) and (5.38). In particular, Theorem 5.10 yields the second relation in (5.41).

Finally we prove (5.42). Since the convergence (5.41) is uniform in t,

$$\lim_{N\to\infty} \inf_{t\in[0,1]} \mathbb{E}\left[Z^{\omega,c}_{\beta_N,h_N}(0,Nt)\right] = \inf_{t\in[0,1]} \mathbf{\Psi}^{\alpha,c}_{C_\alpha\hat{h}}(t) > 0\,,$$

because $t \mapsto \mathbf{\Psi}^{2\alpha-1,c}_{C_\alpha^2\hat{\beta}^2}(t)$ is continuous and strictly positive. On the other hand, for fixed $N \in \mathbb{N}$,

$$\inf_{t\in[0,1]} \mathbb{E}\left[Z^{\omega,c}_{\beta_N,h_N}(0,Nt)\right] = \min_{n\in\{0,1,\dots,N\}} \mathbb{E}\left[Z^{\omega,c}_{\beta_N,h_N}(0,n)\right] > 0\,,$$

so the first relation in (5.42) follows. The second one is proved with analogous arguments. □

Proof of Theorem 5.10. The continuity in t of $\mathbf{\Psi}^{\nu}_{\hat{\delta}}(t), \mathbf{\Psi}^{\nu,c}_{\hat{\delta}}(t)$ can be checked directly by (5.39)-(5.40). They are also non-negative and non-decreasing in $\hat{\delta}$, being pointwise limits of the non-negative and non-decreasing functions (5.37) (these properties are not obviously seen from (5.39)-(5.40)). Since $\mathbf{\Psi}^{\nu}_{\hat{\delta}}(t), \mathbf{\Psi}^{\nu,c}_{\hat{\delta}}(t)$ are clearly analytic functions of $\hat{\delta}$, they must be strictly increasing in $\hat{\delta}$, hence they must be strictly positive, as stated.

Next we prove the convergence results. We focus on the constrained case $\Psi^{c}_{\delta_N}(Nt)$, since the free one is analogous (and simpler). We fix $T \in (0,\infty)$ and show uniform convergence for $t \in [0,T]$. This is equivalent, as one checks by contradiction, to show that for any given sequence $(t_N)_{N\in\mathbb{N}}$ in $[0,T]$ one has $\lim_{N\to\infty} |\Psi^{c}_{\delta_N}(Nt_N) - \mathbf{\Psi}^{\nu,c}_{\hat{\delta}}(t_N)| = 0$. By a subsequence argument, we may assume that $(t_N)_{N\in\mathbb{N}}$ has a limit, say $\lim_{N\to\infty} t_N = t \in [0,T]$, so we are left with proving

$$\lim_{N\to\infty} \Psi^{c}_{\delta_N}(Nt_N) = \mathbf{\Psi}^{\nu,c}_{\hat{\delta}}(t)\,. \tag{5.44}$$

We may safely assume that $Nt_N \in \mathbb{N}_0$, since $\Psi_{\delta_N}(Nt)$ is linearly interpolated for $Nt \notin \mathbb{N}_0$. For notational simplicity we also assume that δ_N is exactly equal to the right hand side of (5.38).

Recalling (5.36), for $0 < n_1 < \ldots < n_k < Nt_N$ we have

$$\mathrm{E}\left[\mathbb{1}_{n_1\in\sigma}\mathbb{1}_{n_2\in\sigma}\cdots\mathbb{1}_{n_k\in\sigma}\,\middle|\, Nt_N \in \sigma\right] = \frac{w(n_1)w(n_2-n_1)\cdots w(Nt_N-n_k)}{w(Nt_N)}\,. \tag{5.45}$$

Since $e^{\delta\mathbb{1}_{n\in\tau}} = 1 + (e^\delta - 1)\mathbb{1}_{n\in\tau}$, a binomial expansion in (5.37) then yields

$$\begin{aligned}\Psi^{\mathrm{c}}_{\delta_N}(Nt_N) &= 1 + \sum_{k=1}^{Nt_N-1}(e^{\delta_N}-1)^k \sum_{0<n_1<\cdots<n_k<Nt_N} \frac{w(n_1)w(n_2-n_1)\cdots w(Nt_N-n_k)}{w(Nt_N)} \\ &= 1 + \sum_{k=1}^{Nt_N-1}\left(\frac{e^{\delta_N}-1}{\delta_N}\right)^k \hat{\delta}^k \left\{\frac{1}{N^k}\sum_{0<n_1<\cdots<n_k<Nt_N}\frac{W_N(0,\frac{n_1}{N})W_N(\frac{n_1}{N},\frac{n_2}{N})\cdots W_N(\frac{n_k}{N},t_N)}{W_N(0,t_N)}\right\},\end{aligned} \tag{5.46}$$

where we have introduced for convenience the rescaled kernel

$$W_N(r,s) := M(N)N^{1-\nu}w(\lceil Ns\rceil - \lceil Nr\rceil)\,, \qquad 0 \le r \le s < \infty\,,$$

and $\lceil x\rceil := \min\{n \in \mathbb{N} : n \ge x\}$ denotes the upper integer part of x. We first show the convergence of the term in brackets in (5.46), for fixed $k \in \mathbb{N}$; later we control the tail of the sum.

For any $\varepsilon > 0$, uniformly for $r - s \ge \varepsilon$ one has $\lim_{N\to\infty} W_N(r,s) = 1/(s-r)^{1-\nu}$, by (5.36). Then, for fixed $k \in \mathbb{N}$, the term in brackets in (5.46) converges to the corresponding integral in (5.40) by a Riemann sum approximation, provided the contribution to the sum given by $n_i - n_{i-1} \le \varepsilon N$ vanishes as $\varepsilon \to 0$, uniformly in $N \in \mathbb{N}$. We show this by a suitable upper bound on $W_N(r,s)$. For any $\eta > 0$, by Potter's bounds [14, Theorem 1.5.6], we have $M(y)/M(x) \le C\max\{(\frac{y}{x})^\eta, (\frac{x}{y})^\eta\}$, hence

$$\frac{C^{-1}}{(r-s)^{1-\nu-\eta}} \le W_N(r,s) \le \frac{C}{(r-s)^{1-\nu+\eta}}, \qquad \forall N \in \mathbb{N},\ \forall 0 \le r \le s \le T\,, \tag{5.47}$$

for some constant $C = C_{\eta,T} < \infty$. Choosing $\eta \in (0,\nu)$, the right hand side in (5.47) is integrable and the contribution to the bracket in (5.46) given by the terms with $n_i - n_{i-1} \le \varepsilon N$ for some i is dominated by the following integral

$$\int\!\!\cdots\!\!\int_{0<t_1<\cdots<t_k<t_N} \frac{C^{k+2}\,t_N^{1-\nu-\eta}}{t_1^{1-\nu+\eta}(t_2-t_1)^{1-\nu+\eta}\cdots(t_N-t_k)^{1-\nu+\eta}}\mathbb{1}_{\{t_i-t_{i-1}\le\varepsilon,\ \text{for some}\ i=1,\cdots,k\}}\prod_{i=1}^k \mathrm{d}t_i\,. \tag{5.48}$$

Plainly, for fixed $k \in \mathbb{N}$, this integral vanishes as $\varepsilon \to 0$ as required (we recall that $t_N \to t < \infty$).

It remains to show that the contribution to (5.46) given by $k \ge M$ can be made small, *uniformly in* $N \in \mathbb{N}$, by taking $M \in \mathbb{N}$ large enough. By (5.47), the term inside the brackets in (5.46) can be bounded from above by the following integral (where we make the change of variables $s_i = t_i/t_N$):

$$\begin{aligned}&\int\!\!\cdots\!\!\int_{0<t_1<\cdots<t_k<t_N} \frac{C^{k+2}\,t_N^{1-\nu-\eta}}{t_1^{1-\nu+\eta}(t_2-t_1)^{1-\nu+\eta}\cdots(t_N-t_k)^{1-\nu+\eta}}\prod_{i=1}^k \mathrm{d}t_i \\ &= \int\!\!\cdots\!\!\int_{0<s_1<\cdots<s_k<1} \frac{C^{k+2}\,t_N^{k(\nu-\eta)-2\eta}}{s_1^{1-\nu+\eta}(s_2-s_1)^{1-\nu+\eta}\cdots(1-s_k)^{1-\nu+\eta}}\prod_{i=1}^k \mathrm{d}s_i \le \hat{C}_T^k\,c_1 e^{-c_2 k\log k}\,,\end{aligned} \tag{5.49}$$

for some constant $\hat{C}_T$ depending only on T (recall that $t_N \to t \in [0,T]$), where the inequality is proved in [23, Lemma B.3], for some constants $c_1, c_2 \in (0,\infty)$, depending only on ν, η. This shows that (5.44) holds and that the limits are finite, completing the proof. □

5.3.2 Proof of relation (5.33)

Assumption (5.10) is equivalent to a suitable concentration inequality for the Euclidean distance $d(x, A) := \inf_{y \in A} |y - x|$ from a point $x \in \mathbb{R}^n$ to a convex set $A \subseteq \mathbb{R}^n$. More precisely, the following Lemma is quite standard (see [63, Proposition 1.3 and Corollary 1.4], except for convexity issues), but for completeness we give a proof in Appendix 5.B.1.

Lemma 5.12. *Assuming* (5.10)*, there exist* $C_1', C_2' \in (0, \infty)$ *such that for every* $n \in \mathbb{N}$ *and for any convex set* $A \subseteq \mathbb{R}^n$ *one has (setting* $\omega = (\omega_1, \ldots, \omega_n)$ *for short)*

$$\mathbb{P}(\omega \in A)\, \mathbb{P}(d(\omega, A) > t) \leq C_1' \exp\left(-\frac{t^\gamma}{C_2'}\right), \qquad \forall t \geq 0\,. \tag{5.50}$$

Viceversa, assuming (5.50)*, relation* (5.10) *holds for suitable* $C_1, C_2 \in (0, \infty)$.

The next result, proved in Appendix 5.B.2, is essentially [63, Proposition 1.6] and shows that (5.50) yields concentration bounds for convex functions that are not necessarily (globally) Lipschitz.

Proposition 5.13. *Assume that* (5.50) *holds for every* $n \in \mathbb{N}$ *and for any convex set* $A \subseteq \mathbb{R}^n$*. Then, for every* $n \in \mathbb{N}$ *and for every differentiable convex function* $f : \mathbb{R}^n \to \mathbb{R}$ *one has*

$$\mathbb{P}(f(\omega) \leq a - t)\, \mathbb{P}(f(\omega) \geq a,\ |\nabla f(\omega)| \leq c) \leq C_1' \exp\left(-\frac{(t/c)^\gamma}{C_2'}\right), \quad \forall a \in \mathbb{R},\ \forall t, c \in (0, \infty)\,, \tag{5.51}$$

where $|\nabla f(\omega)| := \sqrt{\sum_{i=1}^n (\partial_i f(\omega))^2}$ *denotes the Euclidean norm of the gradient of* f.

The usefulness of (5.51) can be understood as follows: given a family of functions $(f_i)_{i \in I}$, if we can control the probabilities $p_i := \mathbb{P}(f_i(\omega) \geq a,\ |\nabla f_i(\omega)| \leq c)$, showing that $\inf_{i \in I} p_i = \theta > 0$ for some fixed a, c, then (5.51) provides a *uniform control on the left tail* $\mathbb{P}(f_i(\omega) \leq a - t)$. This is the key to the proof of relation (5.33), as we now explain.

We recall that $\mathrm{Z}^{\omega, \mathtt{c}}_{\beta_N, h_N}(a, b)$ was defined in (5.30). Our goal is to prove relation (5.33). Some preliminary remarks:

- we consider the case $T = 1$, for notational simplicity;
- we can set $s = 0$ in (5.33), because $\mathrm{Z}^{\omega, \mathtt{c}}_{\beta_N, h_N}(a, b)$ has the same law as $\mathrm{Z}^{\omega, \mathtt{c}}_{\beta_N, h_N}(0, b - a)$.

We can thus reformulate our goal (5.33) as follows: for some constants $A, B < \infty$

$$\sup_{0 \leq t \leq 1} \mathbb{P}\Big(\log \mathrm{Z}^{\omega, \mathtt{c}}_{\beta_N, h_N}(0, Nt) \leq -x\Big) \leq A \exp\left(-\frac{x^\gamma}{B}\right), \qquad \forall x \geq 0,\ \forall N \in \mathbb{N}\,. \tag{5.52}$$

We can further assume that $h_N \leq 0$, because for $h_N > 0$ we have $\mathrm{Z}^{\omega, \mathtt{c}}_{\beta_N, h_N}(0, Nt) \geq \mathrm{Z}^{\omega, \mathtt{c}}_{\beta_N, 0}(0, Nt)$ and replacing h_N by 0 yields a stronger statement. Applying Proposition 5.13 to the functions

$$f_{N,t}(\omega) := \log \mathrm{Z}^{\omega, \mathtt{c}}_{\beta_N, h_N}(0, Nt)\,,$$

relation (5.52) is implied by the following result.

Lemma 5.14. *Fix* $\hat\beta > 0$ *and* $\hat h \leq 0$*. There are constants* $a \in \mathbb{R}$*,* $c \in (0, \infty)$ *such that*

$$\inf_{N \in \mathbb{N}} \inf_{t \in [0,1]} \mathbb{P}(f_{N,t}(\omega) \geq a,\ |\nabla f_{N,t}(\omega)| \leq c) =: \theta > 0\,.$$

Proof. Recall Lemma 5.11, in particular the definition (5.42) of ρ and λ. By the Paley-Zygmund inequality, for all $N \in \mathbb{N}$ and $t \in [0,1]$ we can write

$$\mathbb{P}\left(\mathrm{Z}^{\omega,\mathsf{c}}_{\beta_N,h_N}(0,Nt) \geq \frac{\rho}{2}\right) \geq \mathbb{P}\left(\mathrm{Z}^{\omega,\mathsf{c}}_{\beta_N,h_N}(0,Nt) \geq \frac{\mathbb{E}\left[\mathrm{Z}^{\omega,\mathsf{c}}_{\beta_N,h_N}(0,Nt)\right]}{2}\right) \geq \frac{\left(\mathbb{E}\left[\mathrm{Z}^{\omega,\mathsf{c}}_{\beta_N,h_N}(0,Nt)\right]\right)^2}{4\mathbb{E}\left[\left(\mathrm{Z}^{\omega,\mathsf{c}}_{\beta_N,h_N}(0,Nt)\right)^2\right]}. \tag{5.53}$$

Replacing $h_N \leq 0$ by 0 in the denominator, we get the following lower bound, with $a := \log\frac{\rho}{2}$:

$$\mathbb{P}\left(f_{N,t}(\omega) \geq a\right) = \mathbb{P}\left(\mathrm{Z}^{\omega,\mathsf{c}}_{\beta_N,h_N}(0,Nt) \geq \frac{\rho}{2}\right) \geq \frac{\rho^2}{4\lambda}, \qquad \forall N \in \mathbb{N},\ t \in [0,1]. \tag{5.54}$$

Next we focus on $\nabla f_{N,t}(\omega)$. Recalling (5.30), we have

$$\frac{\partial f_{N,t}}{\partial \omega_i}(\omega) = \beta_N \frac{\mathrm{E}[\mathbb{1}_{i\in\tau} e^{\sum_{k=1}^{Nt-1}(\beta\omega_k - \Lambda(\beta)+h)\mathbb{1}_{k\in\tau}} | Nt \in \tau]}{\mathrm{Z}^{\omega,\mathsf{c}}_{\beta_N,h_N}(0,Nt)} \mathbb{1}_{i\leq Nt-1},$$

hence, denoting by τ' an independent copy of τ,

$$|\nabla f_{N,t}(\omega)|^2 = \sum_{i=1}^N \left(\frac{\partial f_{N,t}}{\partial \omega_i}(\omega)\right)^2 = \beta_N^2 \frac{\mathrm{E}[(\sum_{i=1}^{Nt-1} \mathbb{1}_{i\in\tau\cap\tau'})\, e^{\sum_{k=1}^{Nt-1}(\beta_N\omega_k - \Lambda(\beta_N)+h_N)(\mathbb{1}_{k\in\tau}+\mathbb{1}_{k\in\tau'})} | Nt \in \tau\cap\tau']}{\mathrm{Z}^{\omega,\mathsf{c}}_{\beta_N,h_N}(0,Nt)^2}.$$

Since $h_N \leq 0$, we replace h_N by 0 in the numerator getting an upper bound. Recalling that $a = \log\frac{\rho}{2}$,

$$\begin{aligned}
\mathbb{P}\left(f_{N,t}(\omega) \geq a,\ |\nabla f_{N,t}(\omega)| > c\right) &\leq \frac{\mathbb{E}[|\nabla f_{N,t}(\omega)|^2 \mathbb{1}_{\{f_{N,t}(\omega)\geq a\}}]}{c^2} = \frac{\mathbb{E}[|\nabla f_{N,t}(\omega)|^2 \mathbb{1}_{\{\mathrm{Z}^{\omega,\mathsf{c}}_{\beta_N,h_N}(0,Nt)\geq\frac{\rho}{2}\}}]}{c^2} \\
&\leq \frac{4}{\rho^2c^2}\mathrm{E}\left[\left(\beta_N^2 \sum_{i=1}^{Nt-1} \mathbb{1}_{i\in\tau\cap\tau'}\right) e^{(\Lambda(2\beta_N)-2\Lambda(\beta_N))\sum_{k=1}^{Nt-1}\mathbb{1}_{k\in\tau\cap\tau'}} \middle| Nt \in \tau\cap\tau'\right].
\end{aligned}$$

We recall that $\Lambda(2\beta_N) - 2\Lambda(\beta_N) \sim \beta_N^2$, by (5.2), hence $\Lambda(2\beta_N) - 2\Lambda(\beta_N) \leq C\beta_N^2$ for some $C \in (0,\infty)$. Since $x \leq e^x$ for all $x \geq 0$, we obtain

$$\mathbb{P}\left(f_{N,t}(\omega) \geq a,\ |\nabla f_{N,t}(\omega)| > c\right) \leq \frac{4}{\rho^2c^2}\mathrm{E}\left[e^{(C+1)\beta_N^2\sum_{k=1}^{Nt-1}\mathbb{1}_{k\in\tau\cap\tau'}} \middle| Nt \in \tau\cap\tau'\right] = \frac{4}{\rho^2c^2}\Psi^{\mathsf{c}}_{(C+1)\beta_N^2}(Nt),$$

where we used the definition (5.37), with $\sigma = \tau \cap \tau'$, which we recall that satisfies (5.36) with $\nu = 2\alpha - 1$ and $M(n) = L(n)^2/C_\alpha^2$. In particular, as we discussed in the proof of Lemma 5.11, $\beta_N^2 \sim \delta_N$ in (5.38) with $\hat{\delta} = C_\alpha^2\hat{\beta}^2$, hence $\Psi^{\mathsf{c}}_{(C+1)\beta_N^2}(Nt)$ is uniformly bounded, by Theorem 5.10:

$$\xi := \sup_{N\in\mathbb{N}} \sup_{t\in[0,1]} \Psi^{\mathsf{c}}_{(C+1)\beta_N^2}(Nt) < \infty. \tag{5.55}$$

In conclusion, with ρ, λ, ξ defined in (5.42)-(5.55), setting $a := \log\frac{\rho}{2}$ one has, for every $c > 0$,

$$\begin{aligned}
\mathbb{P}(f_{N,t}(\omega) \geq a,\ |\nabla f_{N,t}(\omega)| \leq c) &= \mathbb{P}(f_{N,t}(\omega) \geq a) - \mathbb{P}(f_{N,t}(\omega) \geq a,\ |\nabla f_{N,t}(\omega)| > c) \\
&\geq \frac{\rho^2}{4\lambda} - \frac{4\xi}{\rho^2c^2} =: \theta, \qquad \forall N \in \mathbb{N},\ t \in [0,1].
\end{aligned}$$

Choosing $c > 0$ large enough one has $\theta > 0$, and the proof is completed. □

5.3.3 Proof of (5.32), case $p \geq 0$.

We recall Garsia's inequality [43] with $\Psi(x) = |x|^p$ and $\phi(u) = u^q$: for all $p \geq 1$, $\mu > 0$ with $p\mu > 4$ we have for every $0 \leq s_i \leq t_i \leq 1$, $i = 1, 2$,

$$\left| Z^{\omega,c}_{\beta_N,h_N}(Ns_1, Nt_1) - Z^{\omega,c}_{\beta_N,h_N}(Ns_2, Nt_2) \right| \leq \frac{8\mu}{\mu - 4/p} B_N \, |(s_1,t_1) - (s_2,t_2)|^{\mu - 4/p} \tag{5.56}$$

where $|\cdot|$ denotes the Euclidean norm and B_N is an explicit (random) constant depending of p:

$$B_N^p = 2^{\mu/2} \int_{[0,1]^2_{\leq} \times [0,1]^2_{\leq}} \frac{\left| Z^{\omega,c}_{\beta_N,h_N}(Ns_1, Nt_1) - Z^{\omega,c}_{\beta_N,h_N}(Ns_2, Nt_2) \right|^p}{|(s_1,t_1) - (s_2,t_2)|^{p\mu}} \mathrm{d}s_1 \mathrm{d}t_1 \mathrm{d}s_2 \mathrm{d}t_2 . \tag{5.57}$$

Since $Z^{\omega,c}_{\beta_N,h_N}(0,0) = 1$ and $|a+b|^p \leq 2^p(|a|^p + |b|^p)$, it follows that

$$\mathbb{E}\left[\sup_{0 \leq s \leq t \leq T} Z^{\omega,c}_{\beta_N,h_N}(Ns, Nt)^p \right] \leq 2^p \left(1 + \left(\frac{8\mu}{\mu - 4/p} \right)^p (\sqrt{2}T)^{p\mu - 4} \mathbb{E}\left[B_N^p \right] \right) .$$

We are thus reduced to estimating $\mathbb{E}[B_N^p]$.

It was shown in [24, Section 2.2] that for any $p \geq 1$ there exist $C_p > 0$ and $\eta_p > 2$ for which

$$\sup_{N \in \mathbb{N}} \mathbb{E}\left(\left| Z^{\omega,c}_{\beta_N,h_N}(Ns_1, Nt_1) - Z^{\omega,c}_{\beta_N,h_N}(Ns_2, Nt_2) \right|^p \right) \leq C_p |(t_1, s_1) - (t_s, s_2)|^{\eta_p} . \tag{5.58}$$

The value of η_p is actually explicit, cf. [24, eq. (2.25), (2.34), last equation in §2.2], and such that

$$\lim_{p \to \infty} \frac{\eta_p}{p} = \bar{\mu} > 0 , \qquad \text{where} \qquad \bar{\mu} = \frac{1}{2} \min\left\{ \alpha' - \tfrac{1}{2}, \delta \right\} ,$$

where $\delta > 0$ is the exponent in (5.9) and α' is any fixed number in $(\frac{1}{2}, \alpha)$. If we choose any $\mu \in (0, \bar{\mu})$, plugging (5.58) into (5.57) we see that the integral is finite for large p, completing the proof. □

5.3.4 Proof of (5.32), case $p \leq 0$.

We prove that an analogue of (5.58) holds. Once proved this, the proof runs as for the case $p \geq 0$, using Garsia's inequality (5.56) for $1/Z^{\omega,c}_{\beta_N,h_N}(Ns, Nt)$.

We first claim that for every $p > 0$ there exists $D_p < \infty$ such that

$$\mathbb{E}\left(Z^{\omega,c}_{\beta_N,h_N}(Ns, Nt)^{-p} \right) \leq D_p , \qquad \forall N \in \mathbb{N}, \; 0 \leq s \leq t \leq 1 . \tag{5.59}$$

This follows by (5.33):

$$\begin{aligned} \mathbb{E}\left(Z^{\omega,c}_{\beta_N,h_N}(0, Nt)^{-p} \right) &= \int_0^\infty \mathbb{P}\left(Z^{\omega,c}_{\beta_N,h_N}(0, Nt)^{-p} > y \right) \mathrm{d}y = \int_0^\infty \mathbb{P}\left(\log Z^{\omega,c}_{\beta_N,h_N}(0, Nt) < -p \log y \right) \mathrm{d}y \\ &\leq 1 + A \int_1^\infty \exp\left(-\frac{p^\gamma (\log y)^\gamma}{B} \right) \mathrm{d}y = 1 + A \int_0^\infty \exp\left(-\frac{p^\gamma x^\gamma}{B} \right) e^x \, \mathrm{d}x < \infty \end{aligned}$$

where in the last step we used $\gamma > 1$. Then, by (5.59), applying the Cauchy-Schwarz inequality twice gives

$$\mathbb{E}\left[\left|\frac{1}{\mathrm{Z}^{\omega,\mathrm{c}}_{\beta_N,h_N}(Ns_1,Nt_1)}-\frac{1}{\mathrm{Z}^{\omega,\mathrm{c}}_{\beta_N,h_N}(Ns_2,Nt_2)}\right|^p\right]=\mathbb{E}\left[\left|\frac{\mathrm{Z}^{\omega,\mathrm{c}}_{\beta_N,h_N}(Ns_1,Nt_1)-\mathrm{Z}^{\omega,\mathrm{c}}_{\beta_N,h_N}(Ns_2,Nt_2)}{\mathrm{Z}^{\omega,\mathrm{c}}_{\beta_N,h_N}(Ns_1,Nt_1)\,\mathrm{Z}^{\omega,\mathrm{c}}_{\beta_N,h_N}(Ns_2,Nt_2)}\right|^p\right]$$

$$\leq \sqrt{D_{4p}}\,\mathbb{E}\left(\left|\mathrm{Z}^{\omega,\mathrm{c}}_{\beta_N,h_N}(Ns_1,Nt_1)-\mathrm{Z}^{\omega,\mathrm{c}}_{\beta_N,h_N}(Ns_2,Nt_2)\right|^{2p}\right)^{\frac{1}{2}} \overset{(5.58)}{\leq} \sqrt{D_{4p}\,C_p}\,|(t_1,s_1)-(t_s,s_2)|^{\eta_{2p}/2}\,,$$

completing the proof. □

5.4 Proof of Theorem 5.6

Throughout this section we fix $\hat\beta > 0$. We recall that the discrete partition functions $\mathrm{Z}^{\omega}_{\beta,h}(Nt)$, $\mathrm{Z}^{\omega,\mathrm{c}}_{\beta,h}(Ns,Nt)$ are linearly interpolated for $Ns, Nt \notin \mathbb{N}_0$. We split the proof in three steps.

Step 1. The coupling. For notational clarity, we denote with the letters $\mathrm{Y}, \mathbf{Y}$ the discrete and continuum partition functions $\mathrm{Z}, \mathbf{Z}$ in which we set $h, \hat h = 0$:

$$\begin{aligned} \mathrm{Y}^{\omega}_{\beta}(N) &:= \mathrm{Z}^{\omega}_{\beta,0}(N)\,, &\qquad \mathbf{Y}^{W}_{\hat\beta}(t) &:= \mathbf{Z}^{W}_{\hat\beta,0}(t)\,,\\ \mathrm{Y}^{\omega,\mathrm{c}}_{\beta}(a,b) &:= \mathrm{Z}^{\omega,\mathrm{c}}_{\beta,0}(a,b)\,, &\qquad \mathbf{Y}^{W,\mathrm{c}}_{\hat\beta}(s,t) &:= \mathbf{Z}^{W,\mathrm{c}}_{\hat\beta,0}(s,t)\,. \end{aligned} \tag{5.60}$$

We know by [24, Theorem 2.1 and Remark 2.3] that for fixed $\hat h$ (in particular, for $\hat h = 0$) the convergence in distribution (5.18), resp. (5.31), holds in the space of continuous functions of $t \in [0,\infty)$, resp. $(s,t) \in [0,\infty)^2_{\leq}$, with uniform convergence on compact sets. By Skorohod's representation theorem (see Remark 5.15 below), we can fix a continuous version of the processes $\mathbf{Y}$ and a coupling of $\mathrm{Y}, \mathbf{Y}$ such that $\mathbb{P}(\mathrm{d}\omega, \mathrm{d}W)$-a.s.

$$\forall T > 0: \quad \sup_{0\leq t\leq T}\left|\mathrm{Y}^{\omega}_{\beta_N}(Nt)-\mathbf{Y}^{W}_{\hat\beta}(t)\right| \xrightarrow[N\to\infty]{} 0\,, \qquad \sup_{0\leq s\leq t\leq T}\left|\mathrm{Y}^{\omega,\mathrm{c}}_{\beta_N}(Ns,Nt)-\mathbf{Y}^{W,\mathrm{c}}_{\hat\beta}(s,t)\right| \xrightarrow[N\to\infty]{} 0\,. \tag{5.61}$$

We stress that the coupling depends only on the fixed value of $\hat\beta > 0$.

The rest of this section consists in showing that under this coupling of $\mathrm{Y}, \mathbf{Y}$, the partition functions converge locally uniformly also in the variable $\hat h$. More precisely, we show that there is a version of the processes $\mathbf{Z}^{W}_{\hat\beta,\hat h}(t)$ and $\mathbf{Z}^{W,\mathrm{c}}_{\hat\beta,\hat h}(s,t)$ such that $\mathbb{P}(\mathrm{d}\omega, \mathrm{d}W)$-a.s.

$$\begin{aligned} \forall T, M \in (0,\infty): \quad &\sup_{0\leq t\leq T,\ |\hat h|\leq M}\left|\mathrm{Z}^{\omega}_{\beta_N,h_N}(Nt)-\mathbf{Z}^{W}_{\hat\beta,\hat h}(t)\right| \xrightarrow[N\to\infty]{} 0\,,\\ &\sup_{0\leq s\leq t\leq T,\ |\hat h|\leq M}\left|\mathrm{Z}^{\omega,\mathrm{c}}_{\beta_N,h_N}(Ns,Nt)-\mathbf{Z}^{W,\mathrm{c}}_{\hat\beta,\hat h}(s,t)\right| \xrightarrow[N\to\infty]{} 0\,. \end{aligned} \tag{5.62}$$

Remark 5.15. A slightly strengthened version of the usual Skorokhod representation theorem [57, Corollaries 5.11–5.12] ensures that one can indeed couple not only the processes $\mathrm{Y}, \mathbf{Y}$, but even the environments ω, W of which they are functions, so that (5.61) holds. More precisely, one can define on the same probability space a Brownian motion W and a family $(\omega^{(N)})_{N\in\mathbb{N}}$, where $\omega^{(N)} = (\omega^{(N)}_i)_{i\in\mathbb{N}}$ is for each N an i.i.d. sequence with the original disorder distribution, such that *plugging* $\omega = \omega^{(N)}$ *into* $\mathrm{Y}^{\omega}_{\beta_N}(\cdot)$, relation (5.61) holds a.s.. (Of course, the sequences $\omega^{(N)}$ and $\omega^{(N')}$ will not be independent for $N \neq N'$.) We write $\mathbb{P}(\mathrm{d}\omega, \mathrm{d}W)$ for the joint probability with respect to $(\omega^{(N)})_{N\in\mathbb{N}}$ and W. For notational simplicity, we will omit the superscript N from $\omega^{(N)}$ in $\mathrm{Y}^{\omega}_{\beta_N}(\cdot)$, $\mathrm{Z}^{\omega}_{\beta_N,h_N}(\cdot)$, etc..

Step 2. Regular versions. The strategy to deduce (5.62) from (5.61) is to express the partition functions $\mathrm{Z}, \mathbf{Z}$ for $\hat h \neq 0$ in terms of the $\hat h = 0$ case, i.e. of $\mathrm{Y}, \mathbf{Y}$. We start doing this in the continuum.

We recall the Wiener chaos expansions of the continuum partition functions, obtained in [23, Theorem 3.1], where as in (5.8) we define the constant $C_\alpha := \frac{\alpha\sin(\alpha\pi)}{\pi}$:

$$\mathbf{Z}^{W}_{\hat\beta,\hat h}(t) = 1 + \sum_{n=1}^{\infty} \int\!\!\cdots\!\!\int_{0<t_1<t_2<\ldots<t_n<t} \frac{C_\alpha^n}{t_1^{1-\alpha}(t_2-t_1)^{1-\alpha}\cdots(t_n-t_{n-1})^{1-\alpha}} \prod_{i=1}^{n} (\hat\beta\,\mathrm{d}W_{t_i} + \hat h\,\mathrm{d}t_i)\,. \tag{5.63}$$

$$\mathbf{Z}^{W,\mathrm{c}}_{\hat\beta,\hat h}(s,t) = 1+ \sum_{n=1}^{\infty} \int\!\!\cdots\!\!\int_{s<t_1<t_2<\ldots<t_n<t} \frac{C_\alpha^n\,(t-s)^{1-\alpha}}{(t_1-s)^{1-\alpha}(t_2-t_1)^{1-\alpha}\cdots(t_n-t_{n-1})^{1-\alpha}(t-t_n)^{1-\alpha}} \prod_{i=1}^{n} (\hat\beta\,\mathrm{d}W_{t_i} + \hat h\,\mathrm{d}t_i)\,. \tag{5.64}$$

These equalities should be understood in the a.s. sense, since stochastic integrals are not defined pathwise. In the next result, of independent interest, we exhibit versions of the continuum partition functions which are jointly continuous in $(t,\hat h)$ and $(s,t,\hat h)$. As a matter of fact, we do not need this result in the sequel, so we only sketch its proof.

Theorem 5.16. *Fix $\hat\beta > 0$ and let $(\mathbf{Y}^{W}_{\hat\beta}(t))_{t\in[0,\infty)}$, $(\mathbf{Y}^{W,\mathrm{c}}_{\hat\beta}(s,t))_{(s,t)\in[0,\infty)^2_\le}$ be versions of* (5.60) *that are continuous in t, resp. in (s,t). Then, for all $\hat h \in \mathbb{R}$ and all $s\in[0,\infty)$, resp. $(s,t)\in[0,\infty)^2_\le$,*

$$\mathbf{Z}^{W}_{\hat\beta,\hat h}(t) \overset{(a.s.)}{=} \mathbf{Y}^{W}_{\hat\beta}(t) + \sum_{k=1}^{\infty} C_\alpha^k\,\hat h^k \left(\int\!\!\cdots\!\!\int_{0<t_1<t_2<\ldots<t_k<t} \frac{\mathbf{Y}^{W,\mathrm{c}}_{\hat\beta}(0,t_1)}{t_1^{1-\alpha}} \frac{\mathbf{Y}^{W,\mathrm{c}}_{\hat\beta}(t_1,t_2)}{(t_2-t_1)^{1-\alpha}} \cdots \frac{\mathbf{Y}^{W,\mathrm{c}}_{\hat\beta}(t_{k-1},t_k)}{(t_k-t_{k-1})^{1-\alpha}} \prod_{i=1}^{k} \mathrm{d}t_i \right), \tag{5.65}$$

$$\begin{aligned}\mathbf{Z}^{W,\mathrm{c}}_{\hat\beta,\hat h}(s,t) &\overset{(a.s.)}{=} \mathbf{Y}^{W,\mathrm{c}}_{\hat\beta}(s,t) \;+\; (t-s)^{1-\alpha}\times \\ &\times \sum_{k=1}^{\infty} C_\alpha^k\,\hat h^k \left(\int\!\!\cdots\!\!\int_{s<t_1<t_2<\ldots<t_k<t} \frac{\mathbf{Y}^{W,\mathrm{c}}_{\hat\beta}(s,t_1)}{(t_1-s)^{1-\alpha}} \frac{\mathbf{Y}^{W,\mathrm{c}}_{\hat\beta}(t_1,t_2)}{(t_2-t_1)^{1-\alpha}} \cdots \frac{\mathbf{Y}^{W,\mathrm{c}}_{\hat\beta}(t_{k-1},t_k)}{(t_k-t_{k-1})^{1-\alpha}} \frac{\mathbf{Y}^{W,\mathrm{c}}_{\hat\beta}(t_k,t)}{(t-t_k)^{1-\alpha}} \prod_{i=1}^{k} \mathrm{d}t_i \right).\end{aligned} \tag{5.66}$$

The right hand sides of (5.65), (5.66) *are versions of the continuum partition functions* (5.63), (5.64) *that are jointly continuous in $(t,\hat h)$, resp. in $(s,t,\hat h)$.*

Remark 5.17. The equalities (5.65) and (5.66) hold on a set of probability 1 which depends on $\hat h$. On the other hand, the right hand sides of these relations are continuous functions of $\hat h$, for W in a fixed set of probability 1.

Proof (sketch). We focus on (5.66), since (5.65) is analogous. We rewrite the n-fold integral in (5.64) expanding the product of differentials in a binomial fashion, obtaining 2^n terms. Each term contains k "deterministic variables" $\mathrm{d}t_i$ and $n-k$ "stochastic variables" $\mathrm{d}W_{t_j}$, whose locations are intertwined. If we relabel the deterministic variables as $u_1 < \ldots < u_k$, performing the sum over n in (5.64) yields

$$\mathbf{Z}^{W,\mathrm{c}}_{\hat\beta,\hat h}(s,t) = 1 + (t-s)^{1-\alpha} \sum_{k=1}^{\infty} C_\alpha^k \int\!\!\cdots\!\!\int_{s<u_1<u_2<\ldots<u_k<t} A(s,u_1)A(u_1,u_2)\cdots A(u_{k-1},u_k)A(u_k,t) \prod_{i=1}^{k} \hat h\,\mathrm{d}u_i\,,$$

where $A(u_m,u_{m+1})$ gathers the contribution of the integrals over the stochastic variables $\mathrm{d}W_{t_j}$ with indexes $t_j \in (u_m, u_{m+1})$, i.e. (relabeling such variables as $t_1, \ldots, t_n$)

$$\begin{aligned} A(a,b) = & \frac{1}{(b-a)^{1-\alpha}} + \\ & + \sum_{n=1}^{\infty} \int \cdots \int_{a<t_1<t_2<\ldots<t_n<b} \frac{C_\alpha^n}{(t_1-a)^{1-\alpha}(t_2-t_1)^{1-\alpha}\cdots(t_n-t_{n-1})^{1-\alpha}(b-t_n)^{1-\alpha}} \prod_{j=1}^{n} \hat\beta \, \mathrm{d}W_{t_j} \, . \end{aligned}$$

A look at (5.64) shows that $A(a,b) = \frac{1}{(b-a)^{1-\alpha}} \mathbf{Z}^{W,\mathrm{c}}_{\hat\beta,0}(s,t) = \frac{1}{(b-a)^{1-\alpha}} \mathbf{Y}^{W,\mathrm{c}}_{\hat\beta}(s,t)$, proving (5.66).

Since the process $\mathbf{Y}^{W,\mathrm{c}}_{\hat\beta}(s,t)$ is continuous by assumption, it is locally bounded and consequently the series in (5.66) converges by the upper bound in [24, Lemma C.1] (that we already used in (5.49)). The continuity of the right hand side of (5.66) in $(s,t,\hat h)$ is then easily checked. □

Step 3. Proof of (5.62). We now prove (5.62), focusing on the second relation, since the first one is analogous. We are going to prove it with $\mathbf{Z}^{W,\mathrm{c}}_{\hat\beta,\hat h}(s,t)$ *defined* as the right hand side of (5.66).

Since $e^{h\mathbb{1}_{n\in\tau}} = 1 + (e^h - 1)\mathbb{1}_{n\in\tau}$, a binomial expansion yields

$$e^{h\sum_{n=q+1}^{r-1}\mathbb{1}_{n\in\tau}} = \prod_{n=q+1}^{r-1} e^{h\mathbb{1}_{n\in\tau}} = 1 + \sum_{k=1}^{r-q-1} \sum_{q+1\le n_1<\cdots<n_k\le r-1} (e^h-1)^k \, \mathbb{1}_{n_1\in\tau}\cdots\mathbb{1}_{n_k\in\tau} \, . \tag{5.67}$$

We now want to plug (5.67) into (5.30). Setting $n_0 := r$, we can write (in analogy with (5.45))

$$\begin{aligned} & \mathrm{E}\left(e^{\sum_{k=q+1}^{r-1}(\beta\omega_k - \Lambda(\beta))\mathbb{1}_{k\in\tau}} \mathbb{1}_{n_1\in\tau}\cdots\mathbb{1}_{n_k\in\tau} \,\middle|\, q\in\tau, r\in\tau \right) \\ & = \left(\prod_{i=1}^{k} e^{\beta\omega_{n_i}-\Lambda(\beta)} \mathrm{Y}^{\omega,\mathrm{c}}_{\beta}(n_{i-1},n_i) \right) \frac{\mathrm{Y}^{\omega,\mathrm{c}}_{\beta}(n_k,r)}{\mathrm{Y}^{\omega,\mathrm{c}}_{\beta}(q,r)} \left(\prod_{i=1}^{k} u(n_i - n_{i-1}) \right) \frac{u(r-n_k)}{u(r-q)} \, , \end{aligned}$$

where we recall that $\mathrm{Y}^{\omega,\mathrm{c}}_{\beta} := \mathrm{Z}^{\omega,\mathrm{c}}_{\beta,0}$, cf.(5.60). For brevity we set

$$Q^{\omega}_{\beta}(a,b) := e^{\beta\omega_a - \Lambda(\beta)} \, \mathrm{Y}^{\omega,\mathrm{c}}_{\beta}(a,b) \, . \tag{5.68}$$

Then, plugging (5.67) into (5.30), we obtain a discrete version of (5.66):

$$\begin{aligned} \mathrm{Z}^{\omega,\mathrm{c}}_{\beta,h}(q,r) = & \mathrm{Y}^{\omega,\mathrm{c}}_{\beta}(q,r) \\ & + \sum_{k=1}^{r-q-1} (e^h-1)^k \sum_{q+1\le n_1<\cdots<n_k\le r-1} \left(\prod_{i=1}^{k} Q(n_{i-1},n_i) \right) \frac{Q^{\omega}_{\beta}(n_k,r)}{Q^{\omega,\mathrm{c}}_{\beta}(q,r)} \left(\prod_{i=1}^{k} u(n_i-n_{i-1}) \right) \frac{u(r-n_k)}{u(r-q)} \, . \end{aligned} \tag{5.69}$$

We are now ready to prove (5.62). For this purpose we are going to use an analogous argument as in Theorem 5.10: it will be necessary and sufficient to prove that, $\mathbb{P}(\mathrm{d}\omega, \mathrm{d}W)$-a.s., for any convergent sequence $(s_N, t_N, \hat h_N)_{N\in\mathbb{N}} \to (s_\infty, t_\infty, \hat h_\infty)$ in $[0,T]^2_\le \times [0,M]$ one has

$$\lim_{N\to\infty} \left| \mathrm{Z}^{\omega,\mathrm{c}}_{\beta_N,\tilde h_N}(Ns_N, Nt_N) - \mathbf{Z}^{W,\mathrm{c}}_{\hat\beta,\hat h_N}(s_N,t_N) \right| = 0 \tag{5.70}$$

where $h_N = \hat h_N \, L(N) N^{-\alpha}$. Recall that we have fixed a coupling under which $\mathrm{Y}^{\omega,\mathrm{c}}_{\beta_N}(Ns,Nt)$ converges uniformly to $\mathbf{Y}^{W,\mathrm{c}}_{\hat\beta}(s,t)$, $\mathbb{P}$-a.s. (cf. (5.61)). Borel-Cantelli estimates ensure that $\max_{a\le N} |\omega_a| = O(\log N)$ $\mathbb{P}$-a.s., by (5.2), hence $Q^{\omega}_{\beta_N}(Ns,Nt)$ also converges uniformly to $\mathbf{Y}^{W,\mathrm{c}}_{\hat\beta}(s,t)$, $\mathbb{P}$-a.s.. We call this event of probability one Ω_{Y} and in the rest of the proof we work on that event, proving (5.70).

It is not restrictive to assume $Ns_N \, Nt_N \in \mathbb{N}_0$. Then we rewrite (5.69) with $q = Ns_N$, $r = Nt_N$ as a

Riemann sum: setting $t_0 = s_N,\ t_{k+1} = t_N$,

$$\begin{aligned} \mathrm{Z}^{\omega,\mathtt{c}}_{\beta_N,h_N}(Ns_N, Nt_N) &= \mathrm{Y}^{\omega,\mathtt{c}}_{\beta_N}(Ns_N, Nt_N) \\ &+ \sum_{k=1}^{N(t_N-s_N)-1} \left(\frac{e^{h_N}-1}{h_N}\right)^k \left\{ \frac{1}{N^k} \sum_{\substack{t_1,\dots,t_k \in \frac{1}{N}\mathbb{N}_0 \\ s_N<t_1<\dots<t_k<t_N}} \frac{\prod_{i=1}^{k+1} \left\{ Q^{\omega}_{\beta_N}(Nt_{i-1}, Nt_i)\,(N\,h_N)\,u(Nt_i - Nt_{i-1}) \right\}}{Q^{\omega,\mathtt{c}}_{\beta_N}(Ns, Nt)\,(N\,h_N)\,u(Nt_N - Ns_N)} \right\}. \end{aligned} \tag{5.71}$$

Observe that $N\,h_N = \hat{h}_N\,L(N)N^{1-\alpha} \sim \hat{h}_\infty\,L(N)N^{1-\alpha}$. Recalling (5.8), on the event Ω_{Y} we have

$$\lim_{N\to\infty} Q^{\omega}_{\beta_N}(Nx, Ny)\,(Nh_N)\,u(\lceil Ny\rceil - \lceil Nx\rceil) = \hat{h}_\infty C_\alpha \frac{\mathbf{Y}^{W,\mathtt{c}}_{\hat{\beta}}(x,y)}{(y-x)^{1-\alpha}} \quad \forall\, 0 \le x < y < \infty, \tag{5.72}$$

and for any $\varepsilon > 0$ the convergence is uniform on $y - x \ge \varepsilon$. Then, for fixed $k \in \mathbb{N}$, the term in brackets in (5.71) converges to the corresponding integral in (5.66), by Riemann sum approximation, because the contribution to the sum given by $t_i - t_{i-1} < \varepsilon$ vanishes as $\varepsilon \to 0$. This claim follows by using Potter's bounds as in (5.47), with $W_N(r,s) = L(N)N^{1-\alpha}u(\lceil Nr\rceil - \lceil Ns\rceil)$, and the uniform convergence of $Q^{\omega}_{\beta_N}(Ns, Nt)$ which provides for any $\eta > 0$ a random constant $C_{\eta,T} \in (0,\infty)$ such that for all $N \in \mathbb{N}$ and for all $0 \le x < y \le T$

$$\frac{C^{-1}_{\eta,T}}{(y-x)^{1-\alpha-\eta}} \le Q^{\omega}_{\beta_N}(Nx, Ny)\,(N\,h_N)\,u(\lceil Ny\rceil - \lceil Nx\rceil) \le \frac{C_{\eta,T}}{(y-x)^{1-\alpha+\eta}}. \tag{5.73}$$

Therefore the contribution of the terms $t_i - t_{i-1} < \varepsilon$ in the brackets of (5.71) is estimated by

$$\int\!\!\cdots\!\!\int_{s_N<t_1<\dots<t_k<t_N} \frac{C^{k+2}_{\eta,T}(s_N - t_N)^{1-\alpha-\eta}}{(t_1 - s_N)^{1-\alpha-\eta}(t_2-t_1)^{1-\alpha+\eta}\cdots(t_N-t_k)^{1-\alpha+\eta}} \mathbb{1}_{\{t_i - t_{i-1}\le\varepsilon,\ \text{for some}\ i=1,\cdots,k\}} \prod_{i=1}^{k} \mathrm{d}t_i.$$

For any fixed $k \in \mathbb{N}$ once chosen $\eta \in (0,\alpha)$ this integral vanishes as $\varepsilon \to 0$ (recall that $(s_N, t_N) \to (s_\infty, t_\infty) \in [0,T]^2_{\le}$). To get the convergence of the whole sum (5.71) we show that the contribution of the terms $k \ge M$ in (5.71) can be made arbitrarily small uniformly in N, by taking M large enough. This follows by the same bound as in (5.49), as the term in brackets in (5.71) is bounded by

$$\begin{aligned} &\int\!\!\cdots\!\!\int_{s_N<t_1<\dots<t_k<t_N} \frac{C^{k+2}_{\eta,T}(s_N - t_N)^{1-\alpha-\eta}}{(t_1 - s_N)^{1-\alpha+\eta}(t_2-t_1)^{1-\alpha+\eta}\cdots(t_N-t_k)^{1-\alpha+\eta}} \mathrm{d}t_1\cdots\mathrm{d}t_k \\ &= \int\!\!\cdots\!\!\int_{0<u_1<\dots<u_k<1} \frac{C^{k+2}_{\eta,T}(t_N - s_N)^{k(\alpha-\eta)-2\eta}}{u_1^{1-\alpha+\eta}(u_2-u_1)^{1-\alpha+\eta}\cdots(1-u_k)^{1-\alpha+\eta}} \mathrm{d}u_1\cdots\mathrm{d}u_k \le (\hat{C}_{\eta,T})^k c_1 e^{-c_2 k\log k}, \end{aligned}$$

for some constant $\hat{C}_{\eta,T} \in (0,\infty)$, cf. [24, Lemma B.3]. This completes the proof. □

5.5 Proof of Theorem 5.4

In this section we prove Theorem 5.4. Most of our efforts are devoted to proving the key relation (5.27), through a fine comparison of the discrete and continuum partition functions, based on a coarse-graining procedure. First of all, we (easily) deduce (5.28) from (5.27).

5.5.1 Proof of relation (5.28) assuming (5.27)

We set $\hat\beta = 1$ and we use (5.17)-(5.19) (with $\varepsilon = \frac{1}{N}$) to rewrite (5.27) as follows: for all $\hat h \in \mathbb{R}$, $\eta > 0$ there exists $\beta_0 > 0$ such that

$$\mathbf{F}^\alpha\left(1, \hat h - \eta\right) \le \frac{\mathrm{F}(\beta,\, \hat h\, \tilde L_\alpha(\frac{1}{\beta})\beta^{\frac{2\alpha}{2\alpha-1}})}{\tilde L_\alpha(\frac{1}{\beta})^2\beta^{\frac{2}{2\alpha-1}}} \le \mathbf{F}^\alpha\left(1, \hat h + \eta\right), \qquad \forall \beta \in (0, \beta_0)\,. \tag{5.74}$$

If we take $\hat h := \mathbf{h}_c^\alpha(1) - 2\eta$, then $\mathbf{F}^\alpha(1, \hat h + \eta) = 0$ by the definition (5.22) of $\mathbf{h}_c^\alpha$. Then (5.74) yields $\mathrm{F}\left(\beta, \hat h\, \tilde L_\alpha(\frac{1}{\beta})\beta^{\frac{2\alpha}{2\alpha-1}}\right) = 0$ for $\beta < \beta_0$, that is $h_c(\beta) \ge \hat h\, \tilde L_\alpha(\frac{1}{\beta})\beta^{\frac{2\alpha}{2\alpha-1}}$ by the definition (5.13) of h_c, hence

$$\liminf_{\beta\to 0} \frac{h_c(\beta)}{\tilde L_\alpha(\frac{1}{\beta})\beta^{\frac{2\alpha}{2\alpha-1}}} \ge \hat h = \mathbf{h}_c^\alpha(1) - 2\eta\,.$$

Letting $\eta \to 0$ proves "half" of (5.28). The other half follows along the same line, choosing $\hat h := \mathbf{h}_c^\alpha(1) + 2\eta$ and using the first inequality in (5.74). □

5.5.2 Renewal process and regenerative set

Henceforth we devote ourselves to the proof of relation (5.27). For $N \in \mathbb{N}$ we consider the *rescaled renewal process*

$$\frac{\tau}{N} = \left\{\frac{\tau_i}{N}\right\}_{i\in\mathbb{N}}$$

viewed as a random subset of $[0,\infty)$. As $N \to \infty$, under the original law P, the random set τ/N converges in distribution to a universal random closed set $\boldsymbol{\tau}^\alpha$, the so-called α-stable regenerative set. We now summarize the few properties of $\boldsymbol{\tau}^\alpha$ that will be needed in the sequel, referring to [24, Appendix A] for more details.

Given a closed subset $C \subseteq \mathbb{R}$ and a point $t \in \mathbb{R}$, we define

$$\mathrm{g}_t(C) := \sup\{x \mid x \in C \cap [-\infty, t)\}\,, \quad \mathrm{d}_t(C) := \inf\{x \mid x \in C \cap [t,\infty)\}\,. \tag{5.75}$$

A key fact is that as $N \to \infty$ the process $((\mathrm{g}_t(\tau/N), \mathrm{d}_t(\tau/N))_{t\in[0,\infty)}$ converges in the sense of finite-dimensional distribution to $((\mathrm{g}_t(\boldsymbol{\tau}^\alpha), \mathrm{d}_t(\boldsymbol{\tau}^\alpha))_{t\in[0,\infty)}$ (see [24, Appendix A]).

Denoting by P_x the law of the regenerative set started at x, that is $\mathrm{P}_x(\boldsymbol{\tau}^\alpha \in \cdot) := \mathrm{P}(\boldsymbol{\tau}^\alpha + x \in \cdot)$, the joint distribution $(\mathrm{g}_t(\boldsymbol{\tau}^\alpha), \mathrm{d}_t(\boldsymbol{\tau}^\alpha))$ is

$$\frac{\mathrm{P}_x\left(\mathrm{g}_t(\boldsymbol{\tau}^\alpha) \in \mathrm{d}u, \mathrm{d}_t(\boldsymbol{\tau}^\alpha) \in \mathrm{d}v\right)}{\mathrm{d}u\,\mathrm{d}v} = C_\alpha \frac{\mathbb{1}_{u\in(x,t)}\mathbb{1}_{v\in(t,\infty)}}{(u-x)^{1-\alpha}(v-u)^{1+\alpha}}\,, \tag{5.76}$$

where $C_\alpha = \frac{\alpha\sin(\pi\alpha)}{\pi}$. We can deduce

$$\frac{\mathrm{P}_x\left(\mathrm{g}_t(\boldsymbol{\tau}^\alpha) \in \mathrm{d}u\right)}{\mathrm{d}u} = \frac{C_\alpha}{\alpha}\frac{\mathbb{1}_{u\in(x,t)}}{(u-x)^{1-\alpha}(t-u)^\alpha}\,, \tag{5.77}$$

$$\frac{\mathrm{P}_x\left(\mathrm{d}_t(\boldsymbol{\tau}^\alpha) \in \mathrm{d}v \mid \mathrm{g}_t(\boldsymbol{\tau}^\alpha) = u\right)}{\mathrm{d}v} = \frac{\alpha\,(t-u)^\alpha}{(v-u)^{1+\alpha}}\mathbb{1}_{v\in(t,\infty)}\,. \tag{5.78}$$

Let us finally state the *regenerative property* of $\boldsymbol{\tau}^\alpha$. Denote by $\mathcal{G}_u$ the filtration generated by $\boldsymbol{\tau}^\alpha \cap [0,u]$ and let σ be a $\{\mathcal{G}_u\}_{u\ge 0}$-stopping time such that $\mathrm{P}(\sigma \in \boldsymbol{\tau}^\alpha) = 1$ (an example is $\sigma = \mathrm{d}_t(\boldsymbol{\tau}^\alpha)$). Then the law of $\boldsymbol{\tau}^\alpha \cap [\sigma,\infty)$ conditionally on $\mathcal{G}_\sigma$ equals $\mathrm{P}_x|_{x=\sigma}$, i.e. the translated random set $(\boldsymbol{\tau}^\alpha - \sigma) \cap [0,\infty)$ is independent of $\mathcal{G}_\sigma$ and it is distributed as the original $\boldsymbol{\tau}^\alpha$ under $\mathrm{P} = \mathrm{P}_0$.

5.5.3 Coarse-grained decomposition

We are going to express the discrete and continuum partition functions in an analogous way, in terms of the random sets τ/N and $\boldsymbol{\tau}^\alpha$, respectively.

We partition $[0,\infty)$ in intervals of length one, called blocks. For a given random set X — it will be either the rescaled renewal process τ/N or the regenerative set $\boldsymbol{\tau}^\alpha$ — we look at the *visited blocks*, i.e. those blocks having non-empty intersection with X. More precisely, we write $[0,\infty) = \bigcup_{k=1}^\infty B_k$, where $B_k = [k-1,k)$, and we say that a block B_k is visited if $X \cap B_k \neq \emptyset$. If we define

$$\mathtt{J}_1(X) := \min\{j > 0 : B_j \cap X \neq \emptyset\}, \qquad \mathtt{J}_k(X) := \min\{j > \mathtt{J}_{k-1} : B_j \cap X \neq \emptyset\}, \tag{5.79}$$

the visited blocks are $(B_{\mathtt{J}_k(X)})_{k\in\mathbb{N}}$. The last visited block before t is $B_{\mathtt{m}_t(X)}$, where we set

$$\mathtt{m}_t(X) := \sup\{k > 0 : \mathtt{J}_k(X) \leq t\}. \tag{5.80}$$

We call $\mathtt{s}_k(X)$ and $\mathtt{t}_k(X)$ the first and last visited points in the block $B_{\mathtt{J}_k(X)}$, i.e. (recalling (5.75))

$$\mathtt{s}_k(X) := \inf\{x \in X \cap B_{\mathtt{J}_k}\} = \mathtt{d}_{\mathtt{J}_k - 1}(X), \qquad \mathtt{t}_k(X) := \sup\{x \in X \cap B_{\mathtt{J}_k}\} = \mathtt{g}_{\mathtt{J}_k}(X). \tag{5.81}$$

(Note that $\mathtt{J}_k(X) = \lfloor \mathtt{s}_k(X) \rfloor = \lfloor \mathtt{t}_k(X) \rfloor$ can be recovered from $\mathtt{s}_k(X)$ or $\mathtt{t}_k(X)$; analogously, $\mathtt{m}_t(X)$ can be recovered from $(\mathtt{J}_k(X))_{k\in\mathbb{N}}$; however, it will be practical to use $\mathtt{J}_k(X)$ and $\mathtt{m}_t(X)$.)

Definition 5.18. The random variables $(\mathtt{J}_k(X), \mathtt{s}_k(X), \mathtt{t}_k(X))_{k\in\mathbb{N}}$ and $(\mathtt{m}_t(X))_{t\in\mathbb{N}}$ will be called the *coarse-grained decomposition* of the random set $X \subseteq [0,\infty)$. In case $X = \boldsymbol{\tau}^\alpha$ we will simply write $(\boldsymbol{\mathtt{J}}_k, \boldsymbol{\mathtt{s}}_k, \boldsymbol{\mathtt{t}}_k)_{k\in\mathbb{N}}$ and $(\boldsymbol{\mathtt{m}}_t)_{t\in\mathbb{N}}$, while in case $X = \tau/N$ we will write $(\mathtt{J}_k^{(N)}, \mathtt{s}_k^{(N)}, \mathtt{t}_k^{(N)})_{k\in\mathbb{N}}$ and $(\mathtt{m}_t^{(N)})_{t\in\mathbb{N}}$.

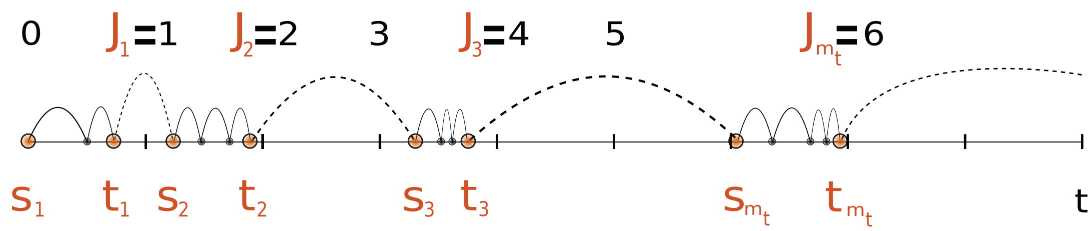

Fig. 5.1: In the figure we have pictured a random set X, given as the zero level set of a stochastic process, whose excursions are represented by the semi-arcs (dotted arcs represents excursions between two consecutive visited blocks). The coarse-grained decomposition of X is given by the first and last points – $\mathtt{s}_k(X), \mathtt{t}_k(X)$ – inside each visited block $[\mathtt{J}_k - 1(X), \mathtt{J}_k(X))$, marked by a big dot in the figure. By construction, between visited blocks there are no points of X; all of its points are contained in the set $\cup_{k\in\mathbb{N}}\big[\mathtt{s}_k(X), \mathtt{t}_k(X)\big]$.

Remark 5.19. For every $t \in \mathbb{N}$, one has the convergence in distribution

$$\left(\mathtt{m}_t^{(N)}, (\mathtt{s}_k^{(N)}, \mathtt{t}_k^{(N)})_{1\leq k\leq \mathtt{m}_t^{(N)}}\right) \xrightarrow[N\to\infty]{d} \left(\boldsymbol{\mathtt{m}}_t, (\boldsymbol{\mathtt{s}}_k, \boldsymbol{\mathtt{t}}_k)_{1\leq k\leq \boldsymbol{\mathtt{m}}_t}\right), \tag{5.82}$$

thanks to the convergence in distribution of $(\mathtt{g}_s(\tau/N), \mathtt{d}_s(\tau/N))_{s\in\mathbb{N}}$ toward $(\mathtt{g}_s(\boldsymbol{\tau}^\alpha), \mathtt{d}_s(\boldsymbol{\tau}^\alpha))_{s\in\mathbb{N}}$.

Using (5.76) and the regenerative property, one can write explicitly the joint density of $\boldsymbol{\mathtt{J}}_k, \boldsymbol{\mathtt{s}}_k, \boldsymbol{\mathtt{t}}_k$. This yields the following estimates of independent interest, proved in Appendix 5.A.1.

Lemma 5.20. *For any $\alpha \in (0,1)$ there are constants $A_\alpha, B_\alpha \in (0,\infty)$ such that for all $\gamma \geq 0$*

$$\sup_{(x,y)\in[0,1]^2_\leq} \mathrm{P}_x\left(\mathtt{t}_2 \in [\mathtt{J}_2 - \gamma, \mathtt{J}_2] \,|\, \mathtt{t}_1 = y\right) \leq A_\alpha\, \gamma^{1-\alpha}\,, \tag{5.83}$$

$$\sup_{(x,y)\in[0,1]^2_\leq} \mathrm{P}_x\left(\mathtt{t}_2 - \mathtt{s}_2 \leq \gamma \,|\, \mathtt{t}_1 = y\right) \leq B_\alpha\, \gamma^{\alpha}\,, \tag{5.84}$$

where P_x *is the law of the α-stable regenerative set starting from x.*

We are ready to express the partition functions $\mathrm{Z}^{\omega}_{\beta_N,h_N}(Nt)$ and $\mathbf{Z}^{W}_{\hat\beta,\hat h}(t)$ in terms of the random sets τ/N and $\boldsymbol{\tau}^\alpha$, through their coarse-grained decompositions. Recall that β_N, h_N are linked to N and $\hat\beta, \hat h$ by (5.17). For notational lightness, we denote by E the expectation with respect to either τ/N or $\boldsymbol{\tau}^\alpha$.

Theorem 5.21 (Coarse-grained Hamiltonians)**.** *For $t \in \mathbb{N}$ we can write the discrete and continuum partition functions as follows:*

$$\mathrm{Z}^{\omega}_{\beta_N,h_N}(Nt) = \mathrm{E}\left[e^{\mathrm{H}^{\omega}_{N,t;\hat\beta,\hat h}(\tau/N)}\right]\,, \qquad \mathbf{Z}^{W}_{\hat\beta,\hat h}(t) = \mathrm{E}\left[e^{\mathbf{H}^{W}_{t;\hat\beta,\hat h}(\boldsymbol{\tau}^\alpha)}\right]\,, \tag{5.85}$$

where the coarse-grained Hamiltonians $\mathrm{H}(\tau/N)$ *and* $\mathbf{H}(\boldsymbol{\tau}^\alpha)$ *depend on the random sets τ/N and $\boldsymbol{\tau}^\alpha$ only through their coarse-grained decompositions, and are defined by*

$$\mathrm{H}^{\omega}_{N,t;\hat\beta,\hat h}(\tau/N) := \sum_{k=1}^{\mathtt{m}_t^{(N)}} \log \mathrm{Z}^{\omega,\mathtt{c}}_{\beta_N,h_N}(N\mathtt{s}_k^{(N)}, N\mathtt{t}_k^{(N)})\,, \qquad \mathbf{H}^{W}_{t;\hat\beta,\hat h}(\boldsymbol{\tau}^\alpha) = \sum_{k=1}^{\mathtt{m}_t} \log \mathbf{Z}^{W,\mathtt{c}}_{\hat\beta,\hat h}(\mathtt{s}_k, \mathtt{t}_k)\,. \tag{5.86}$$

Proof. Starting from the definition (5.7) of $\mathrm{Z}^{\omega}_{\beta_N,h_N}(Nt)$, we disintegrate according to the random variables $\mathtt{m}_t^{(N)}$ and $(\mathtt{s}_k^{(N)}, \mathtt{t}_k^{(N)})_{1\leq k\leq \mathtt{m}_t^{(N)}}$. Recalling (5.30), the renewal property of τ yields

$$\mathrm{Z}^{\omega}_{\beta_N,h_N}(Nt) = \mathrm{E}\left[\mathrm{Z}^{\omega,\mathtt{c}}_{\beta_N,h_N}(0, N\mathtt{t}_1^{(N)})\, \mathrm{Z}^{\omega,\mathtt{c}}_{\beta_N,h_N}(N\mathtt{s}_2^{(N)}, N\mathtt{t}_2^{(N)}) \cdots \mathrm{Z}^{\omega,\mathtt{c}}_{\beta_N,h_N}(N\mathtt{s}^{(N)}_{\mathtt{m}_t^{(N)}}, N\mathtt{t}^{(N)}_{\mathtt{m}_t^{(N)}})\right]\,, \tag{5.87}$$

which is precisely the first relation in (5.85), with H defined as in (5.86).

The second relation in (5.85) can be proved with analogous arguments, by the regenerative property of $\boldsymbol{\tau}^\alpha$. Alternatively, one can exploit the convergence in distribution (5.82), that becomes a.s. convergence under a suitable coupling of τ/N and $\boldsymbol{\tau}^\alpha$; since $\mathrm{Z}^{\omega,\mathtt{c}}_{\beta_N,h_N}(Ns, Nt) \to \mathbf{Z}^{W,\mathtt{c}}_{\hat\beta,\hat h}(s,t)$ uniformly for $0 \leq s \leq t \leq T$, under a coupling of ω and W (by Theorem 5.6), letting $N \to \infty$ in (5.87) yields, by dominated convergence, the second relation in (5.85), with $\mathbf{H}$ defined as in (5.86). □

The usefulness of the representations in (5.85) is that they express the discrete and continuum partition functions in closely analogous ways, which behave well in the continuum limit $N \to \infty$. To appreciate this fact, note that although the discrete partition function is expressed through an Hamiltonian of the form $\sum_{n=1}^{N}(\beta\omega_n - \Lambda(\beta) + h)\mathbb{1}_{\{n\in\tau\}}$, cf. (5.7), such a "microscopic" Hamiltonian admits no continuum analogue, because the continuum disordered pinning model studied in [24] is *singular* with respect to the regenerative set $\boldsymbol{\tau}^\alpha$, cf. [24, Theorem 1.5]. The "macroscopic" coarse-grained Hamiltonians in (5.86), on the other hand, will serve our purpose.

5.5.4 General Strategy

We now describe a general strategy to prove the key relation (5.27) of Theorem 5.4, exploiting the representations in (5.85). We follow the strategy developed for the copolymer model in [16, 22], with some simplifications and strengthenings.

Definition 5.22. Let $f_t(N, \hat\beta, \hat h)$ and $g_t(N, \hat\beta, \hat h)$ be two real functions of $t, N \in \mathbb{N}$, $\hat\beta > 0$, $\hat h \in \mathbb{R}$. We write

$f \prec g$ if for all fixed $\hat\beta, \hat h, \hat h'$ with $\hat h < \hat h'$ there exists $N_0(\hat\beta, \hat h, \hat h') < \infty$ such that for all $N > N_0$

$$\begin{aligned} \limsup_{t\to\infty} f_t(N,\hat\beta,\hat h) &\le \limsup_{t\to\infty} g_t(N,\hat\beta,\hat h'), \\ \liminf_{t\to\infty} f_t(N,\hat\beta,\hat h) &\le \liminf_{t\to\infty} g_t(N,\hat\beta,\hat h'). \end{aligned} \tag{5.88}$$

where the limits are taken along $t \in \mathbb{N}$. If both $f \prec g$ and $g \prec f$ hold, then we write $f \simeq g$.

Keeping in mind (5.11) and (5.20), we define $f^{(1)}$ and $f^{(3)}$ respectively as the continuum and discrete (rescaled) finite-volume free energies, averaged over the disorder:

$$f_t^{(1)}(N,\hat\beta,\hat h) := \frac{1}{t}\mathbb{E}\left(\log \mathbf{Z}^W_{\hat\beta,\hat h}(t)\right), \tag{5.89}$$

$$f_t^{(3)}(N,\hat\beta,\hat h) := \frac{1}{t}\mathbb{E}\left(\log Z^\omega_{\beta_N,h_N}(Nt)\right). \tag{5.90}$$

(Note that $f^{(1)}$ does not depend on N.) *Our goal is to prove that* $f^{(3)} \simeq f^{(1)}$, because this yields the key relation (5.27) in Theorem 5.4, and also the existence of the averaged continuum free energy as $t \to \infty$ along $t \in \mathbb{N}$ (thus proving part of Theorem 5.3). Let us start checking these claims.

Lemma 5.23. *Assuming $f^{(3)} \simeq f^{(1)}$, the following limit exists along $t \in \mathbb{N}$ and is finite:*

$$\mathbf{F}^\alpha(\hat\beta,\hat h) := \lim_{t\to\infty} f_t^{(1)}(N,\hat\beta,\hat h) = \lim_{t\to\infty}\frac{1}{t}\mathbb{E}\left(\log \mathbf{Z}^W_{\hat\beta,\hat h}(t)\right). \tag{5.91}$$

Proof. The key point is that $f_t^{(3)}$ admits a limit as $t \to \infty$: by (5.11), for all $N \in \mathbb{N}$ we can write

$$\lim_{t\to\infty} f_t^{(3)}(N,\hat\beta,\hat h) = N\,\mathrm{F}(\beta_N, h_N) \tag{5.92}$$

where we agree that limits are taken along $t \in \mathbb{N}$. For every $\varepsilon > 0$, the relation $f^{(3)} \simeq f^{(1)}$ yields

$$\limsup_{t\to\infty} f_t^{(1)}(N,\hat\beta,\hat h - 2\varepsilon) \le \lim_{t\to\infty} f_t^{(3)}(N,\hat\beta,\hat h - \varepsilon) \le \liminf_{t\to\infty} f_t^{(1)}(N,\hat\beta,\hat h), \tag{5.93}$$

for $N \in \mathbb{N}$ large enough (depending on $\hat\beta, \hat h$ and ε). Plugging the definition (5.89) of $f_t^{(1)}$, which does not depend on $N \in \mathbb{N}$, into this relation, we get

$$\limsup_{t\to\infty}\frac{1}{t}\mathbb{E}\left(\log \mathbf{Z}^W_{\hat\beta,\hat h-2\varepsilon}(t)\right) \le \liminf_{t\to\infty}\frac{1}{t}\mathbb{E}\left(\log \mathbf{Z}^W_{\hat\beta,\hat h}(t)\right). \tag{5.94}$$

The left hand side of this relation is a convex function of $\varepsilon \ge 0$ (being the $\limsup$ of convex functions, by Proposition 5.7) and is finite (it is bounded by $N\,\mathrm{F}(\beta_N, h_N) < \infty$, by (5.92) and (5.93)). It follows that it is a continuous function of $\varepsilon \ge 0$, so letting $\varepsilon \downarrow 0$ completes the proof. □

Lemma 5.24. *Assuming $f^{(3)} \simeq f^{(1)}$, relation* (5.27) *in Theorem 5.4 holds true.*

Proof. We know that $\lim_{t\to\infty} f_t^{(1)}(N,\hat\beta,\hat h) = \mathbf{F}^\alpha(\hat\beta,\hat h)$ by Lemma 5.23. Recalling (5.92), relation $f^{(3)} \simeq f^{(1)}$ can be restated as follows: for all $\hat\beta > 0$, $\hat h \in \mathbb{R}$ and $\eta > 0$ there exists $N_0 < \infty$ such that

$$\mathbf{F}^\alpha(\hat\beta,\hat h - \eta) \le N\,\mathrm{F}\left(\hat\beta\frac{L(N)}{N^{\alpha-\frac12}}, \hat h\frac{L(N)}{N^\alpha}\right) \le \mathbf{F}^\alpha(\hat\beta,\hat h + \eta), \qquad \forall N \ge N_0.$$

Incidentally, this relation holds also when $N \in [N_0,\infty)$ is not an integer, because the same holds for relation (5.92). Setting $\varepsilon := \frac{1}{N}$ and $\varepsilon_0 := \frac{1}{N_0}$ yields precisely relation (5.27). □

The rest of this section is devoted to proving $f^{(1)} \simeq f^{(3)}$. By (5.89)-(5.90) and (5.85), we can write

$$f_t^{(1)}(N,\hat{\beta},\hat{h}) = \frac{1}{t}\mathbb{E}\left(\log \mathrm{E}\left[e^{\mathbf{H}^W_{t;\hat{\beta},\hat{h}}(\boldsymbol{\tau}^\alpha)}\right]\right), \qquad f_t^{(3)}(N,\hat{\beta},\hat{h}) = \frac{1}{t}\mathbb{E}\left(\log \mathrm{E}\left[e^{\mathrm{H}^\omega_{N,t;\hat{\beta},\hat{h}}(\tau/N)}\right]\right). \tag{5.95}$$

Since relation $\simeq$ is transitive, it suffices to prove that

$$f^{(1)} \simeq f^{(2)} \simeq f^{(3)}, \tag{5.96}$$

for a suitable intermediate quantity $f^{(2)}$ which somehow interpolates between $f^{(1)}$ and $f^{(3)}$. We define $f^{(2)}$ replacing the rescaled renewal τ/N by the regenerative set $\boldsymbol{\tau}^\alpha$ in $f^{(3)}$:

$$f_t^{(2)}(N,\hat{\beta},\hat{h}) := \frac{1}{t}\mathbb{E}\left(\log \mathrm{E}\left[e^{\mathrm{H}^\omega_{N,t;\hat{\beta},\hat{h}}(\boldsymbol{\tau}^\alpha)}\right]\right). \tag{5.97}$$

Note that each function $f^{(i)}$, for $i = 1,2,3$, is of the form

$$f_t^{(i)}(N,\hat{\beta},\hat{h}) = \frac{1}{t}\mathbb{E}\left(\log \mathrm{E}\left[e^{\mathrm{H}^{(i)}_{N,t;\hat{\beta},\hat{h}}}\right]\right), \tag{5.98}$$

for a suitable Hamiltonian $\mathrm{H}^{(i)}_{N,t;\hat{\beta},\hat{h}}$. We recall that $\mathbb{E}$ is expectation with respect to the disorder (either ω or W) while E is expectation with respect to the random set (either τ/N or $\boldsymbol{\tau}^\alpha$).

The general strategy to to prove $f^{(i)} \prec f^{(j)}$ can be described as follows ($i = 1$, $j = 2$ for clarity). For fixed $\hat{\beta},\hat{h},\hat{h}'$ with $\hat{h} < \hat{h}'$, we couple the two Hamiltonians $\mathrm{H}^{(1)}_{N,t;\hat{\beta},\hat{h}}$ and $\mathrm{H}^{(2)}_{N,t;\hat{\beta},\hat{h}'}$ (both with respect to the random set and to the disorder) and we define for $\varepsilon \in (0,1)$

$$\Delta^{(1,2)}_{N,\varepsilon}(t) := \mathrm{H}^{(1)}_{N,t;\hat{\beta},\hat{h}} - (1-\varepsilon)\mathrm{H}^{(2)}_{N,\hat{\beta},\hat{h}'} \tag{5.99}$$

(we omit the dependence of $\Delta^{(1,2)}_{N,\varepsilon}(t)$ on $\hat{\beta},\hat{h},\hat{h}'$ for short). Hölder's inequality then gives

$$\mathrm{E}\left(e^{\mathrm{H}^{(1)}_{N,t;\hat{\beta},\hat{h}}}\right) \le \mathrm{E}\left(e^{\mathrm{H}^{(2)}_{N,t;\hat{\beta},\hat{h}'}}\right)^{1-\varepsilon} \mathrm{E}\left(e^{\frac{1}{\varepsilon}\Delta^{(1,2)}_{N,\varepsilon}(t)}\right)^{\varepsilon}.$$

Denoting by $\lim^*_{t\to\infty}$ either $\liminf_{t\to\infty}$ or $\limsup_{t\to\infty}$ (or, for that matter, the limit of any convergent subsequence), recalling (5.98) and applying Jensen's inequality leads to

$$\overset{*}{\lim_{t\to\infty}} f_t^{(1)}(N,\hat{\beta},\hat{h}) \le (1-\varepsilon)\overset{*}{\lim_{t\to\infty}} f_t^{(2)}(N,\hat{\beta},\hat{h}') + \varepsilon \limsup_{t\to\infty} \frac{1}{t}\log \mathrm{E}\mathbb{E}\left(e^{\frac{1}{\varepsilon}\Delta^{(1,2)}_{N,\varepsilon}(t)}\right).$$

In order to prove $f^{(1)} \prec f^{(2)}$ it then suffices to show the following: for fixed $\hat{\beta},\hat{h},\hat{h}'$ with $\hat{h} < \hat{h}'$,

$$\exists \varepsilon \in (0,1),\ N_0 \in (0,\infty): \qquad \limsup_{t\to\infty} \frac{1}{t}\log \mathrm{E}\mathbb{E}\left(e^{\frac{1}{\varepsilon}\Delta^{(1,2)}_{N,\varepsilon}(t)}\right) \le 0, \quad \forall N \ge N_0. \tag{5.100}$$

(Of course, ε and N_0 will depend on the fixed values of $\hat{\beta},\hat{h},\hat{h}'$.)

We will give details only for the proof of $f^{(1)} \prec f^{(2)} \prec f^{(3)}$, because with analogous arguments one proves $f^{(1)} \succ f^{(2)} \succ f^{(3)}$. Before starting, we describe the coupling of the coarse-grained Hamiltonians.

Remark 5.25. For technical convenience, instead of linearly interpolating the discrete partition functions when $Ns, Nt \notin \mathbb{N}_0$, it will be convenient in §5.5.7 to consider their piecewise constant extension $\mathrm{Z}^{\omega,\mathtt{c}}_{\beta_N,h_N}(\lfloor Ns\rfloor,\lfloor Nt\rfloor)$. Plainly, relation (5.62) still holds.

5.5.5 The coupling

The coarse-grained Hamiltonians H and $\mathbf{H}$, defined in (5.86), are functions of the disorders ω and W and of the random sets τ/N and $\boldsymbol{\tau}^\alpha$. We now describe how to couple the disorders (the random sets will be coupled through Radon-Nikodym derivatives, cf. §5.5.7).

Recall that $[a,b)^2_\le := \{(x,y): a \le x \le y \le b\}$. For $n \in \mathbb{N}$, we let $\mathrm{Z}_N^{(n)}$ and $\mathbf{Z}^{(n)}$ denote the families of discrete and continuum partition functions with endpoints in $[n, n+1)$:

$$\mathrm{Z}_N^{(n)} := \Big(\mathrm{Z}^{\omega,\mathrm{c}}_{\beta_N,h_N}(Ns,Nt)\Big)_{(s,t)\in[n,n+1)^2_\le}, \qquad \mathbf{Z}^{(n)} := \Big(\mathbf{Z}^{W,\mathrm{c}}_{\hat\beta,\hat h}(s,t)\Big)_{(s,t)\in[n,n+1)^2_\le}.$$

Note that both $(\mathrm{Z}_N^{(n)})_{n\in\mathbb{N}}$ and $(\mathbf{Z}^{(n)})_{n\in\mathbb{N}}$ are i.i.d. sequences. A look at (5.86) reveals that that the coarse-grained Hamiltonian H depends on the disorder ω only through $(\mathrm{Z}_N^{(n)})_{n\in\mathbb{N}}$, and likewise $\mathbf{H}$ depends on W only through $(\mathbf{Z}^{(n)})_{n\in\mathbb{N}}$. Consequently, to couple H and $\mathbf{H}$ it suffices to couple $(\mathrm{Z}_N^{(n)})_{n\in\mathbb{N}}$ and $(\mathbf{Z}^{(n)})_{n\in\mathbb{N}}$, i.e. to define a law for the joint sequence $((\mathrm{Z}_N^{(n)}, \mathbf{Z}^{(n)}))_{n\in\mathbb{N}}$. *We take this to be i.i.d.*: discrete and continuum partition functions are coupled independently in each block $[n, n+1)$.

It remains to define a coupling for $\mathrm{Z}_N^{(1)}$ and $\mathbf{Z}^{(1)}$. Throughout the sequel we fix $\hat\beta > 0$ and $\hat h, \hat h' \in \mathbb{R}$ with $\hat h < \hat h'$. We can then use the coupling provided by Theorem 5.6, which ensures that relation (5.62) holds $\mathbb{P}(\mathrm{d}\omega, \mathrm{d}W)$-a.s., with $T = 1$ and $M = \max\{|\hat h|, |\hat h'|\}$.

5.5.6 First step: $f^{(1)} \prec f^{(2)}$

Our goal is to prove (5.100). Recalling (5.99), (5.95) and (5.97), as well as (5.86), for fixed $\hat\beta, \hat h, \hat h'$ with $\hat h < \hat h'$ we can write

$$\Delta^{(1,2)}_{N,\varepsilon}(t) = \mathbf{H}^W_{t;\hat\beta,\hat h}(\boldsymbol{\tau}^\alpha) - (1-\varepsilon)\mathrm{H}^W_{N,t;\hat\beta,\hat h'}(\boldsymbol{\tau}^\alpha) = \sum_{k=1}^{\mathtt{m}_t} \log \frac{\mathbf{Z}^{W,\mathrm{c}}_{\hat\beta,\hat h}(\mathtt{s}_k,\mathtt{t}_k)}{\mathrm{Z}^{\omega,\mathrm{c}}_{\beta_N,h'_N}(N\mathtt{s}_k, N\mathtt{t}_k)^{1-\varepsilon}}, \tag{5.101}$$

where we set $h'_N = \hat h' L(N)/N^\alpha$ for short, cf. (5.17). Consequently

$$\mathrm{E}\mathbb{E}\left(e^{\frac{1}{\varepsilon}\Delta^{(1,2)}_{N,\varepsilon}(t)}\right) = \mathrm{E}\left[\prod_{k=1}^{\mathtt{m}_t} f_{N,\varepsilon}(\mathtt{s}_k,\mathtt{t}_k)\right], \qquad \text{where} \quad f_{N,\varepsilon}(s,t) := \mathbb{E}\left[\left(\frac{\mathbf{Z}^{W,\mathrm{c}}_{\hat\beta,\hat h}(s,t)}{\mathrm{Z}^{\omega,\mathrm{c}}_{\beta_N,h'_N}(Ns,Nt)^{1-\varepsilon}}\right)^{\frac{1}{\varepsilon}}\right], \tag{5.102}$$

because discrete and continuum partition functions are coupled independently in each block $[n, n+1)$, cf. §5.5.5, hence the $\mathbb{E}$-expectation factorizes. (Of course, $f_{N,\varepsilon}(s,t)$ also depends on $\hat\beta, \hat h, \hat h'$.)

Let us denote by $\mathcal{F}_M = \sigma((\mathtt{s}_i,\mathtt{t}_i) : i \le M)$ the filtration generated by the first M visited blocks. By the regenerative property, the regenerative set $\boldsymbol{\tau}^\alpha$ starts afresh at the stopping time $\mathtt{s}_{k-1}$, hence

$$\mathrm{E}[f_{N,\varepsilon}(\mathtt{s}_k,\mathtt{t}_k)\,|\,\mathcal{F}_{k-1}] = \mathrm{E}[f_{N,\varepsilon}(\mathtt{s}_k,\mathtt{t}_k)\,|\,\mathtt{s}_{k-1},\mathtt{t}_{k-1}], \tag{5.103}$$

where we agree that $\mathrm{E}[\,\cdot\,|\,\mathtt{s}_0,\mathtt{t}_0] := \mathrm{E}[\,\cdot\,]$. Defining the constant

$$\Lambda_{N,\varepsilon} := \sup_{k,\mathtt{s}_{k-1},\mathtt{t}_{k-1}} \mathrm{E}[f_{N,\varepsilon}(\mathtt{s}_k,\mathtt{t}_k)\,|\,\mathtt{s}_{k-1},\mathtt{t}_{k-1}], \tag{5.104}$$

we have $\mathrm{E}[f_{N,\varepsilon}(\mathtt{s}_k,\mathtt{t}_k)\,|\,\mathcal{F}_{k-1}] \le \Lambda_{N,\varepsilon}$, hence $\mathrm{E}\left[\prod_{k=1}^M f_{N,\varepsilon}(\mathtt{s}_k,\mathtt{t}_k)\right] \le (\Lambda_{N,\varepsilon})^M$ for every $M \in \mathbb{N}$, hence

$$\mathrm{E}\mathbb{E}\left(e^{\frac{1}{\varepsilon}\Delta^{(1,2)}_{N,\varepsilon}(t)}\right) = \mathrm{E}\left[\prod_{k=1}^{\mathtt{m}_t} f_{N,\varepsilon}(\mathtt{s}_k,\mathtt{t}_k)\right] \le \sum_{M=1}^\infty \mathrm{E}\left[\prod_{k=1}^{M} f_{N,\varepsilon}(\mathtt{s}_k,\mathtt{t}_k)\right] \le \sum_{M=1}^\infty (\Lambda_{N,\varepsilon})^M = \frac{\Lambda_{N,\varepsilon}}{1-\Lambda_{N,\varepsilon}} < \infty, \tag{5.105}$$

provided $\Lambda_{N,\varepsilon} < 1$. The next Lemma shows that this is indeed the case, if $\varepsilon > 0$ is small enough and $N > N_0(\varepsilon)$. This completes the proof of (5.100), hence of $f^{(1)} \prec f^{(2)}$.

Lemma 5.26. *The following relation holds for* $\Lambda_{N,\varepsilon}$ *defined in* (5.104)*, with* $f_{N,\varepsilon}$ *defined in* (5.102)*:*

$$\limsup_{\varepsilon\to 0}\, \limsup_{N\to\infty} \Lambda_{N,\varepsilon} = 0. \tag{5.106}$$

The proof of Lemma 5.26 is deferred to the Appendix 5.A.2. The key idea is that, for fixed $s < t$, the function $f_{N,\varepsilon}(s,t)$ in (5.102) is small when $\varepsilon > 0$ small and N large, because the discrete partition function in the denominator is close to the continuum one appearing in the numerator, but with $\hat{h}' > \hat{h}$ (recall that the continuum partition function is strictly increasing in $\hat{h}$, by Proposition 5.7). To prove that $\Lambda_{N,\varepsilon}$ in (5.104) is small, we replace s, t by the random points $\mathbf{s}_k, \mathbf{t}_k$, showing that they cannot be too close to each other, conditionally on (and uniformly over) $\mathbf{s}_{k-1}, \mathbf{t}_{k-1}$.

5.5.7 Second Step: $f^{(2)} \prec f^{(3)}$

Recalling (5.95) and (5.85)-(5.86), we can write $f^{(3)}$ as follows:

$$f_t^{(3)}(N, \hat{\beta}, \hat{h}) = \frac{1}{t}\mathbb{E}\left(\log \mathrm{E}\left[\prod_{k=1}^{\mathtt{m}_t^{(N)}} \mathrm{Z}^{\omega,\mathtt{c}}_{\beta_N,h_N}(N\mathtt{s}_k^{(N)}, N\mathtt{t}_k^{(N)})\right]\right). \tag{5.107}$$

Note that $f^{(2)}$, defined in (5.97), enjoys the same representation (5.107), with $\mathtt{m}_t^{(N)}$ and $\mathtt{s}_k^{(N)}, \mathtt{t}_k^{(N)}$ replaced respectively by their continuum counterparts $\mathbf{m}_t$ and $\mathbf{s}_k, \mathbf{t}_k$. Since we extend the discrete partition function in a piecewise constant fashion $\mathrm{Z}^{\omega,\mathtt{c}}_{\beta_N,h_N}(\lfloor Ns \rfloor, \lfloor Nt \rfloor)$, cf. Remark 5.25, we can replace $\mathbf{s}_k, \mathbf{t}_k$ by their left neighbors $\mathbf{s}_k^{(N)}, \mathbf{t}_k^{(N)}$ on the lattice $\frac{1}{N}\mathbb{N}_0$, i.e.

$$\mathbf{s}_k^{(N)} := \frac{\lfloor N\mathbf{s}_k \rfloor}{N}, \qquad \mathbf{t}_k^{(N)} := \frac{\lfloor N\mathbf{t}_k \rfloor}{N}, \tag{5.108}$$

getting to the following representation for $f_t^{(2)}$:

$$f_t^{(2)}(N, \hat{\beta}, \hat{h}) = \frac{1}{t}\mathbb{E}\left(\log \mathrm{E}\left[\prod_{k=1}^{\mathbf{m}_t} \mathrm{Z}^{\omega,\mathtt{c}}_{\beta_N,h_N}(N\mathbf{s}_k^{(N)}, N\mathbf{t}_k^{(N)})\right]\right). \tag{5.109}$$

The random vectors $(\mathbf{m}_t, (\mathbf{s}_k^{(N)}, \mathbf{t}_k^{(N)})_{1 \le k \le \mathbf{m}_t^{(N)}})$ and $(\mathtt{m}_t^{(N)}, (\mathtt{s}_k^{(N)}, \mathtt{t}_k^{(N)})_{1 \le k \le \mathtt{m}_t^{(N)}})$ are mutually absolutely continuous. Let us denote by R_t the Radon-Nikodym derivative

$$\mathrm{R}_t\left(M, (x_k, y_k)_{k=1}^M\right) = \frac{\mathrm{P}\left(\mathtt{m}_t^{(N)} = M, (\mathtt{s}_k^{(N)}, \mathtt{t}_k^{(N)})_{k=1}^m = (x_k, y_k)_{k=1}^M\right)}{\mathrm{P}\left(\mathbf{m}_t = M, (\mathbf{s}_k^{(N)}, \mathbf{t}_k^{(N)})_{k=1}^M = (x_k, y_k)_{k=1}^M\right)}, \tag{5.110}$$

for $M \in \mathbb{N}$ and $x_k, y_k \in \frac{1}{N}\mathbb{N}_0$ (note that necessarily $x_1 = 0$). We can then rewrite (5.107) as follows:

$$f_t^{(3)}(N, \hat{\beta}, \hat{h}) = \frac{1}{t}\mathbb{E}\left(\log \mathrm{E}\left[\prod_{k=1}^{\mathbf{m}_t} \left(\mathrm{Z}^{\omega,\mathtt{c}}_{\beta_N,h_N}(N\mathbf{s}_k^{(N)}, N\mathbf{t}_k^{(N)})\right) \cdot \mathrm{R}_t\left(\mathbf{m}_t, (\mathbf{s}_k^{(N)}, \mathbf{t}_k^{(N)})_{k=1}^{\mathbf{m}_t}\right)\right]\right), \tag{5.111}$$

which is identical to (5.109), apart from the Radon-Nikodym derivative R_t.

Relations (5.109) and (5.111) are useful because $f_t^{(2)}$ and $f_t^{(3)}$ are averaged with respect to *the same random set* $\boldsymbol{\tau}^\alpha$ (through its coarse-grained decomposition $\mathbf{m}_t$ and $\mathbf{s}_k^{(N)}, \mathbf{t}_k^{(N)}$). This allows to apply the general strategy of §5.5.4. Defining $\Delta_{N,\varepsilon} = \Delta_{N,\varepsilon}^{(2,3)}$ as in (5.99), we can write by (5.109)-(5.111)

$$\mathrm{E}\mathbb{E}\left(e^{\frac{1}{\varepsilon}\Delta_{N,\varepsilon}(t)}\right) = \mathrm{E}\left[\left\{\prod_{k=1}^{\mathbf{m}_t} \mathbb{E}\left[\left(\frac{\mathrm{Z}^{\omega,\mathtt{c}}_{\beta_N,h_N}(N\mathbf{s}_k^{(N)}, N\mathbf{t}_k^{(N)})}{\mathrm{Z}^{\omega,\mathtt{c}}_{\beta_N,h'_N}(N\mathbf{s}_k^{(N)}, N\mathbf{t}_k^{(N)})^{1-\varepsilon}}\right)^{\frac{1}{\varepsilon}}\right]\right\} \mathrm{R}_t\left(\mathbf{m}_t, (\mathbf{s}_k^{(N)}, \mathbf{t}_k^{(N)})_{k=1}^{\mathbf{m}_t}\right)^{\frac{1}{\varepsilon}}\right], \tag{5.112}$$

and our goal is to prove (5.100) with $\Delta_{N,\varepsilon}^{(1,2)}$ replaced by $\Delta_{N,\varepsilon}$: explicitly, for fixed $\hat{\beta}, \hat{h}, \hat{h}'$ with $\hat{h} < \hat{h}'$,

$$\exists \varepsilon \in (0,1),\ N_0 \in (0,\infty): \qquad \limsup_{t\to\infty} \frac{1}{t} \log \mathrm{E}\mathbb{E}\left(e^{\frac{1}{\varepsilon}\Delta_{N,\varepsilon}(t)}\right) \le 0, \quad \forall N \ge N_0. \tag{5.113}$$

In order to simplify (5.112), in analogy with (5.102), we define

$$g_{N,\varepsilon}(s,t) := \mathbb{E}\left[\left(\frac{Z^{\omega,c}_{\beta_N,h_N}(Ns,Nt)}{Z^{\omega,c}_{\beta_N,h'_N}(Ns,Nt)^{1-\varepsilon}}\right)^{\frac{1}{\varepsilon}}\right]. \tag{5.114}$$

The Radon-Nikodym derivative R_t in (5.110) does not factorize exactly, but an approximate factorization holds: as we show in section 5.A.3 (cf. Lemma 5.29), for suitable functions r_N and $\tilde{r}_N$

$$\mathrm{R}_t\left(M,(x_k,y_k)_{k=1}^M\right) \le \left\{\prod_{\ell=1}^{M} r_N(y_{\ell-1},x_\ell,y_\ell)\right\}\tilde{r}_N(y_M,t), \tag{5.115}$$

where we set $y_0 := 0$ (also note that $x_1 = 0$). Looking back at (5.112), we can write

$$\mathrm{E}\mathbb{E}\left(e^{\frac{1}{\varepsilon}\Delta_{N,\varepsilon}(t)}\right) \le \mathrm{E}\left[\left\{\prod_{k=1}^{\mathtt{m}_t} g_{N,\varepsilon}\left(\mathbf{s}_k^{(N)},\mathbf{t}_k^{(N)}\right) r_N\left(\mathbf{t}_{k-1}^{(N)},\mathbf{s}_k^{(N)},\mathbf{t}_k^{(N)}\right)^{\frac{1}{\varepsilon}}\right\}\tilde{r}_N\left(\mathbf{t}_{\mathtt{m}_t}^{(N)},t\right)^{\frac{1}{\varepsilon}}\right]. \tag{5.116}$$

Let us now explain the strategy. We can easily get rid of the last term $\tilde{r}_N$ by Cauchy-Schwarz, so we focus on the product appearing in brackets. The goal would be to prove that (5.113) holds by bounding (5.116) through a geometric series, as in (5.105). This could be obtained, in analogy with (5.103)-(5.104), by showing that for ε small and N large the conditional expectation

$$\mathrm{E}\left[g_{N,\varepsilon}(\mathbf{s}_k^{(N)},\mathbf{t}_k^{(N)})\,r_N(\mathbf{t}_{k-1}^{(N)},\mathbf{s}_k^{(N)},\mathbf{t}_k^{(N)})^{\frac{1}{\varepsilon}}\,\middle|\,\mathcal{F}_{k-1}\right] = \mathrm{E}\left[g_{N,\varepsilon}(\mathbf{s}_k^{(N)},\mathbf{t}_k^{(N)})\,r_N(\mathbf{t}_{k-1}^{(N)},\mathbf{s}_k^{(N)},\mathbf{t}_k^{(N)})^{\frac{1}{\varepsilon}}\,\middle|\,\mathbf{s}_{k-1},\mathbf{t}_{k-1}\right]$$

is smaller than 1, *uniformly in* $\mathbf{s}_{k-1},\mathbf{t}_{k-1}$. Unfortunately this fails, because the Radon-Nikodym term r_N is *not* small when $\mathbf{t}_{k-1}$ is close to the right end of the block to which it belongs, i.e. to $\mathtt{J}_{k-1}$.

To overcome this difficulty, we distinguish the two events $\{\mathbf{t}_{k-1} \le \mathtt{J}_{k-1}-\gamma\}$ and $\{\mathbf{t}_{k-1} > \mathtt{J}_{k-1}-\gamma\}$, for $\gamma > 0$ that will be chosen small enough. The needed estimates on the functions $g_{N,\varepsilon}$, r_N and $\tilde{r}_N$ are summarized in the next Lemma, proved in Appendix 5.A.3. Let us define for $p \ge 1$ the constant

$$\Lambda_{N,\varepsilon,p} := \sup_{k,\mathbf{s}_{k-1},\mathbf{t}_{k-1}} \mathrm{E}\left(g_{N,\varepsilon}(\mathbf{s}_k,\,\mathbf{t}_k)^p\,\middle|\,\mathbf{s}_{k-1},\mathbf{t}_{k-1}\right), \tag{5.117}$$

where we recall that $g_{N,\varepsilon}(s,t)$ is defined in (5.114), and we agree that $\mathrm{E}[\,\cdot\,|\,\mathbf{s}_0,\mathbf{t}_0] := \mathrm{E}[\,\cdot\,]$.

Lemma 5.27. *Let us fix $\hat{\beta} \in \mathbb{R}$ and $\hat{h},\hat{h}' \in \mathbb{R}$ with $\hat{h} < \hat{h}'$.*

- *For all $p \ge 1$*

$$\limsup_{\varepsilon\to 0}\,\limsup_{N\to\infty}\,\Lambda_{N,\varepsilon,p} = 0. \tag{5.118}$$

- *For all $\varepsilon \in (0,1)$, $p \ge 1$ there is $C_{\varepsilon,p} < \infty$ such that for all $N \in \mathbb{N}$*

$$\forall k \ge 2: \quad \mathrm{E}\left[r_N\left(\mathbf{t}_{k-1}^{(N)},\,\mathbf{s}_k^{(N)},\,\mathbf{t}_k^{(N)}\right)^{\frac{p}{\varepsilon}}\,\middle|\,\mathbf{s}_{k-1}\,\mathbf{t}_{k-1}\right] \le C_{\varepsilon,p}, \tag{5.119}$$

$$\mathrm{E}\left[\tilde{r}_N\left(\mathbf{t}_{\mathtt{m}_t}^{(N)},t\right)^{\frac{p}{\varepsilon}}\right] \le C_{\varepsilon,p}. \tag{5.120}$$

- *For all $\varepsilon \in (0,1)$, $p \ge 1$, $\gamma \in (0,1)$ there is $\tilde{N}_0 = \tilde{N}_0(\varepsilon,\,p,\,\gamma) < \infty$ such that for $N \ge \tilde{N}_0$*

$$\forall k \ge 2: \quad \mathrm{E}\left[r_N\left(\mathbf{t}_{k-1}^{(N)},\,\mathbf{s}_k^{(N)},\,\mathbf{t}_k^{(N)}\right)^{\frac{p}{\varepsilon}}\,\middle|\,\mathbf{s}_{k-1}\,\mathbf{t}_{k-1}\right] \le 2 \quad \text{on the event } \{\mathbf{t}_{k-1} \le \mathtt{J}_{k-1}-\gamma\}, \tag{5.121}$$

$$\mathrm{E}\left[r_N\left(0,\,0,\,\mathbf{t}_1^{(N)}\right)^{\frac{p}{\varepsilon}}\right] \le 2. \tag{5.122}$$

We are ready to estimate (5.116), with the goal of proving (5.113). Let us define

$$\Phi_{k,N}^{(\varepsilon)} := g_{N,\varepsilon}\left(\mathtt{s}_k^{(N)},\, \mathtt{t}_k^{(N)}\right)^2\, r_N\left(\mathtt{t}_{k-1}^{(N)},\, \mathtt{s}_k^{(N)},\, \mathtt{t}_k^{(N)}\right)^{\frac{2}{\varepsilon}}, \tag{5.123}$$

with the convention that $\mathtt{t}_0^{(N)} := 0$ (note that also $\mathtt{s}_1^{(N)} = 0$). Then, by (5.120) and Cauchy-Schwarz,

$$\mathrm{E}\mathbb{E}\left(e^{\frac{1}{\varepsilon}\Delta_{N,\varepsilon}(t)}\right) \le C_{\varepsilon,2}\,\mathrm{E}\left[\prod_{k=1}^{\mathtt{m}_t} \Phi_{k,N}^{(\varepsilon)}\right] \le C_{\varepsilon,2} \sum_{M=1}^{\infty} \mathrm{E}\left[\prod_{k=1}^{M} \Phi_{k,N}^{(\varepsilon)}\right].$$

We are going to show that

$$\exists \varepsilon \in (0,1),\ N_0 \in (0,\infty): \qquad \mathrm{E}\left[\prod_{k=1}^{M} \Phi_{k,N}^{(\varepsilon)}\right] \le \frac{1}{2^M} \quad \forall M \in \mathbb{N},\ N \ge N_0, \tag{5.124}$$

which yields the upper bound $\mathrm{E}\mathbb{E}(e^{\frac{1}{\varepsilon}\Delta_{N,\varepsilon}(t)}) \le C_{\varepsilon,2}$, completing the proof of (5.113).

In the next Lemma, that will be proved in a moment, we single out some properties of $\Phi_{k,N}^{(\varepsilon)}$, that are direct consequence of Lemma 5.27.

Lemma 5.28. *One can choose $\varepsilon \in (0,1)$, $c \in (1,\infty)$, $\gamma \in (0,1)$ and $N_0 < \infty$ such that for $N \ge N_0$*

$$\mathrm{E}\left[\Phi_{1,N}^{(\varepsilon)}\right] \le \frac{1}{4}; \qquad \forall k \ge 2: \quad \mathrm{E}\left[\Phi_{k,N}^{(\varepsilon)} \,\middle|\, \mathtt{s}_{k-1},\, \mathtt{t}_{k-1}\right] \le \begin{cases} c & \text{always} \\ \frac{1}{4} & \text{on } \{\mathtt{t}_{k-1} \le \mathtt{J}_{k-1} - \gamma\} \end{cases}, \tag{5.125}$$

and moreover

$$\mathrm{E}\left[\Phi_{1,N}^{(\varepsilon)} \mathbb{1}_{\{\mathtt{t}_1 > 1-\gamma\}}\right] \le \frac{1}{8c}; \qquad \forall k \ge 2: \quad \mathrm{E}\left[\Phi_{k,N}^{(\varepsilon)} \mathbb{1}_{\{\mathtt{t}_k > \mathtt{J}_k - \gamma\}} \,\middle|\, \mathtt{s}_{k-1},\, \mathtt{t}_{k-1}\right] \le \frac{1}{8c}. \tag{5.126}$$

Let us now deduce (5.124). We fix ε, c, γ and N_0 as in Lemma 5.28. Setting for compactness

$$D_{M,N} := \prod_{k=1}^{M} \Phi_{k,N}^{(\varepsilon)},$$

we show the following strengthened version of (5.124):

$$\mathrm{E}\left[D_{M,N}\right] \le \frac{1}{2^M}, \qquad \mathrm{E}\left[D_{M,N} \mathbb{1}_{\{\mathtt{t}_M > \mathtt{J}_M - \gamma\}}\right] \le \frac{1}{c 2^{M+2}}, \qquad \forall M \in \mathbb{N},\ N \ge N_0. \tag{5.127}$$

We proceed by induction on $M \in \mathbb{N}$. The case $M = 1$ holds by the first relations in (5.125), (5.126). For the inductive step, we fix $M \ge 2$ and we assume that (5.127) holds for $M - 1$, then

$$\begin{aligned}
\mathrm{E}\left[D_{M,N}\right] &= \mathrm{E}\left[D_{M-1,N}\,\mathrm{E}\left(\Phi_{M,N}^{(\varepsilon)} \,\middle|\, \mathcal{F}_{M-1}\right)\right] = \mathrm{E}\left[D_{M-1,N}\,\mathrm{E}\left(\Phi_{M,N}^{(\varepsilon)} \,\middle|\, \mathtt{s}_{M-1},\, \mathtt{t}_{M-1}\right)\right] \\
&= \mathrm{E}\left[D_{M-1,N}\,\mathrm{E}\left(\Phi_{M,N}^{(\varepsilon)} \,\middle|\, \mathtt{s}_{M-1},\, \mathtt{t}_{M-1}\right) \mathbb{1}_{\{\mathtt{t}_{M-1} > \mathtt{J}_{M-1} - \gamma\}}\right] \\
&\quad + \mathrm{E}\left[D_{M-1,N}\,\mathrm{E}\left(\Phi_{M,N}^{(\varepsilon)} \,\middle|\, \mathtt{s}_{M-1},\, \mathtt{t}_{M-1}\right) \mathbb{1}_{\{\mathtt{t}_{M-1} \le \mathtt{J}_{M-1} - \gamma\}}\right] \\
&\le c\,\mathrm{E}\left[D_{M-1,N} \mathbb{1}_{\{\mathtt{t}_{M-1} > \mathtt{J}_{M-1} - \gamma\}}\right] + \frac{1}{4}\,\mathrm{E}\left[D_{M-1,N}\right] \le c\,\frac{1}{c 2^{M+1}} + \frac{1}{4}\frac{1}{2^{M-1}} \le \frac{1}{2^M},
\end{aligned}$$

where in the last line we have applied (5.125) and the induction step. Similarly, applying the second relation in (5.126) and the induction step,

$$\mathrm{E}\left[D_{M,N} \mathbb{1}_{\{\mathtt{t}_M > \mathtt{J}_M - \gamma\}}\right] = \mathrm{E}\left[D_{M-1,N}\mathrm{E}\left(\Phi_{M,N}^{(\varepsilon)} \mathbb{1}_{\{\mathtt{t}_M > \mathtt{J}_M - \gamma\}} \,\middle|\, \mathtt{s}_{M-1},\, \mathtt{t}_{M-1}\right)\right] \le \frac{1}{8c}\mathrm{E}\left[D_{M-1,N}\right] \le \frac{1}{c 2^{M+2}}.$$

This completes the proof of (5.127), hence of (5.124), hence of $f^{(2)} \prec f^{(3)}$.

Proof of Lemma 5.28. We fix $\varepsilon > 0$ such that, by relation (5.118), for some $\hat{N}_0 < \infty$ one has

$$\Lambda_{N,\varepsilon,4p} \le \frac{1}{32}, \quad \forall N \ge \hat{N}_0, \quad \text{for both } p = 1 \text{ and } p = 2. \tag{5.128}$$

Given the parameter $\gamma \in (0,1)$, to be fixed later, we are going to apply relations (5.121)-(5.122), that hold for $N \ge \tilde{N}_0(\gamma)$ and for $p \in \{1,2\}$ (we stress that ε has been fixed). Defining $N_0 := \max\{\tilde{N}_0(\gamma), \hat{N}_0\}$, whose value will be fixed once γ is fixed, henceforth we assume that $N \ge N_0$.

Recalling (5.123) and (5.117), for $k \ge 2$ and $p \in \{1,2\}$ one has, by Cauchy-Schwarz,

$$\begin{aligned}
\mathrm{E}\left[\left(\Phi_{k,N}^{(\varepsilon)}\right)^p \,\middle|\, \mathtt{s}_{k-1},\, \mathtt{t}_{k-1}\right]^2 &\le \mathrm{E}\left[g_{N,\varepsilon}\left(\mathtt{s}_k^{(N)},\, \mathtt{t}_k^{(N)}\right)^{4p} \,\middle|\, \mathtt{s}_{k-1},\, \mathtt{t}_{k-1}\right] \cdot \\
&\quad \cdot \mathrm{E}\left[r_N\left(\mathtt{t}_{k-1}^{(N)},\, \mathtt{s}_k^{(N)},\, \mathtt{t}_k^{(N)}\right)^{\frac{4p}{\varepsilon}} \,\middle|\, \mathtt{s}_{k-1},\, \mathtt{t}_{k-1}\right] \\
&\le \Lambda_{N,\varepsilon,4p} \cdot \mathrm{E}\left[r_N\left(\mathtt{t}_{k-1}^{(N)},\, \mathtt{s}_k^{(N)},\, \mathtt{t}_k^{(N)}\right)^{\frac{4p}{\varepsilon}} \,\middle|\, \mathtt{s}_{k-1},\, \mathtt{t}_{k-1}\right] \\
&\le \begin{cases} \frac{1}{32} \cdot C_{\varepsilon,4p} & \text{always} \\ \frac{1}{32} \cdot 2 = \frac{1}{4^2} & \text{on } \{\mathtt{t}_{k-1} \le \mathtt{J}_{k-1} - \gamma\} \end{cases}
\end{aligned} \tag{5.129}$$

having used (5.128). Setting $p = 1$, the second relation in (5.125) holds with $c := \sqrt{\frac{C_{\varepsilon,4}}{32}}$. The first relation in (5.125) is proved similarly, setting $\mathrm{E}[\,\cdot\,|\,\mathtt{s}_0, \mathtt{t}_0] := \mathrm{E}[\,\cdot\,]$ in (5.129) and applying (5.122).

Coming to (5.126), by Cauchy-Schwarz

$$\begin{aligned}
\mathrm{E}\left[\Phi_{k,N}^{(\varepsilon)} \mathbb{1}_{\{\mathtt{t}_k > \mathtt{J}_k - \gamma\}} \,\middle|\, \mathtt{s}_{k-1},\, \mathtt{t}_{k-1}\right]^2 &\le \mathrm{E}\left[\left(\Phi_{k,N}^{(\varepsilon)}\right)^2 \,\middle|\, \mathtt{s}_{k-1},\, \mathtt{t}_{k-1}\right] \cdot \mathrm{P}\left(\mathtt{t}_k > \mathtt{J}_k - \gamma \,|\, \mathtt{s}_{k-1}, \mathtt{t}_{k-1}\right) \\
&\le \frac{C_{\varepsilon,8}}{32}\left\{\sup_{(x,y)\in[0,1)^2_{\le}} \mathrm{P}_x\left(\mathtt{t}_2 > \mathtt{J}_2 - \gamma \,|\, \mathtt{t}_1 = y\right)\right\},
\end{aligned} \tag{5.130}$$

having applied (5.129) for $p = 2$, together with the regenerative property and translation invariance of $\boldsymbol{\tau}^\alpha$. By Lemma 5.20, we can choose $\gamma > 0$ small enough so that the second relation in (5.126) holds (recall that $c > 1$ has already been fixed, as a function of ε only). The first relation in (5.126) holds by similar arguments, setting $\mathrm{E}[\,\cdot\,|\,\mathtt{s}_0, \mathtt{t}_0] := \mathrm{E}[\,\cdot\,]$ in (5.130). □

5.6 Proof of Theorem 5.3

The existence and finiteness of the limit (5.20) has been already proved in Lemma 5.23. The fact that $\mathbf{F}^\alpha(\hat{\beta}, \hat{h})$ is non-negative and convex in $\hat{h}$ follows immediately by relation (5.26) (which is a consequence of Theorem 5.4, that we have already proved), because the discrete partition function $\mathrm{F}(\beta, h)$ has these properties. (Alternatively, one could also give direct proofs of these properties, following the same path as for the discrete model.) Finally, the scaling relation (5.21) holds because $\mathbf{Z}^W_{\hat{\beta},\hat{h}}(ct)$ has the same law as $\mathbf{Z}^W_{c^{\alpha-\frac{1}{2}}\hat{\beta},c^\alpha\hat{h}}(t)$, by (5.17)-(5.18) (see also [24, Theorem 2.4]).

5.A Regenerative Set

5.A.1 Proof of Lemma 5.20

We may safely assume that $\gamma < \frac{1}{4}$, since for $\gamma \ge \frac{1}{4}$ relations (5.83)-(5.84) are trivially satisfied, by choosing A_α, B_α large enough.

We start by (5.84), partitioning on the index $\mathtt{J}_2$ of the block containing $\mathtt{s}_2$, $\mathtt{t}_2$ (recall (5.79), (5.81)):

$$\mathrm{P}_x\left(\mathtt{t}_2 - \mathtt{s}_2 \le \gamma \mid \mathtt{t}_1 = y\right) = \sum_{n=2}^{\infty} \mathrm{P}_x\left(\mathtt{t}_2 - \mathtt{s}_2 \le \gamma,\ \mathtt{J}_2 = n \mid \mathtt{t}_1 = y\right),$$

for $(x, y) \in [0, 1]^2_{\le}$. Then (5.84) is proved if we show that there exists $c_\alpha \in (0, \infty)$ such that

$$p_n(\gamma, x, y) := \mathrm{P}_x\left(\mathtt{t}_2 - \mathtt{s}_2 \le \gamma,\ \mathtt{J}_2 = n \mid \mathtt{t}_1 = y\right) \le \frac{c_\alpha}{n^{1+\alpha}}\,\gamma^\alpha, \quad \forall n \ge 2,\ \forall (x, y) \in [0, 1]^2_{\le}. \tag{5.131}$$

Let us write down the density of $(\mathtt{t}_2, \mathtt{s}_2, \mathtt{J}_2)$ given $\mathtt{s}_1 = x$, $\mathtt{t}_1 = y$. Writing for simplicity $\mathtt{g}_t := \mathtt{g}_t(\boldsymbol{\tau}^\alpha)$ and $\mathtt{d}_t := \mathtt{d}_t(\boldsymbol{\tau}^\alpha)$, we can write for $(z, w) \in [n-1, n]^2_{\le}$

$$\begin{aligned}\mathrm{P}_x\left(\mathtt{s}_2 \in \mathrm{d}z\,,\mathtt{t}_2 \in \mathrm{d}w\,,\mathtt{J}_2 = n \mid \mathtt{t}_1 = y\right) &= \frac{\mathrm{P}_x\left(\mathtt{g}_1 \in \mathrm{d}y,\ \mathtt{d}_1 \in \mathrm{d}z,\ \mathtt{g}_n \in \mathrm{d}w\right)}{\mathrm{P}_x(\mathtt{g}_1 \in \mathrm{d}y)} \\ &= \frac{\mathrm{P}_x\left(\mathtt{g}_1 \in \mathrm{d}y,\ \mathtt{d}_1 \in \mathrm{d}z\right)\mathrm{P}_z\left(\mathtt{g}_n \in \mathrm{d}w\right)}{\mathrm{P}_x(\mathtt{g}_1 \in \mathrm{d}y)}\,,\end{aligned}$$

where we have applied the regenerative property at the stopping time $\mathtt{d}_1$. Then by (5.76), (5.77) we get

$$\begin{aligned}\frac{\mathrm{P}_x\left(\mathtt{s}_2 \in \mathrm{d}z\,,\mathtt{t}_2 \in \mathrm{d}w\,,\mathtt{J}_2 = n \mid \mathtt{t}_1 = y\right)}{\mathrm{d}z\,\mathrm{d}w} &= C_\alpha\,\frac{(1-y)^\alpha}{(z-y)^{1+\alpha}(w-z)^{1-\alpha}(n-w)^\alpha} \\ &\text{for} \quad x \le y \le 1\,,\ \ n-1 \le z \le w \le n\,.\end{aligned} \tag{5.132}$$

where $C_\alpha = \frac{\alpha\sin(\pi\alpha)}{\pi}$. Note that this density is independent of x. Integrating over w, by (5.77) we get

$$\frac{\mathrm{P}_x\left(\mathtt{s}_2 \in \mathrm{d}z\,,\mathtt{J}_2 = n \mid \mathtt{t}_1 = y\right)}{\mathrm{d}z} = \alpha\,\frac{(1-y)^\alpha}{(z-y)^{1+\alpha}} \qquad \text{for} \quad x \le y \le 1\,,\ \ n-1 \le z \le n\,. \tag{5.133}$$

We can finally estimate $p_n(\gamma, x, y)$ in (5.131). We compute separately the contributions from the events $\{\mathtt{s}_2 \le n - \gamma\}$ and $\{\mathtt{s}_2 > n - \gamma\}$, starting with the former. By (5.132)

$$\begin{aligned}&C_\alpha\,(1-y)^\alpha \int_{n-1}^{n-\gamma} \frac{1}{(z-y)^{1+\alpha}}\left(\int_z^{z+\gamma} \frac{1}{(w-z)^{1-\alpha}(n-w)^\alpha}\,\mathrm{d}w\right)\mathrm{d}z \\ &\quad\le \frac{C_\alpha}{\alpha}\,(1-y)^\alpha\,\gamma^\alpha \int_{n-1}^{n-\gamma} \frac{1}{(z-y)^{1+\alpha}}\,\frac{1}{(n-\gamma-z)^\alpha}\mathrm{d}z\,,\end{aligned} \tag{5.134}$$

because $n - w \ge n - \gamma - z$. In case $n \ge 3$, since $z - y \ge n - 2$ (recall that $y \in [0, 1]$),

$$(5.134) \le \frac{C_\alpha}{\alpha}\,\gamma^\alpha\,\frac{1}{(n-2)^{1+\alpha}} \int_{n-1}^{n-\gamma} \frac{1}{(n-\gamma-z)^\alpha}\,\mathrm{d}z \le \frac{C_\alpha}{\alpha(1-\alpha)}\,\frac{\gamma^\alpha}{(n-2)^{1+\alpha}}\,, \tag{5.135}$$

which matches with the right hand side of (5.131) (just estimate $n - 2 \ge n/3$ for $n \ge 3$). The same computation works also for $n = 2$, provided we restrict the last integral in (5.134) on $\frac{3}{2} \le z \le 2 - \gamma$, which leads to (5.135) with $(n-2)$ replaced by $1/2$. On the other hand, in case $n = 2$ and $1 \le z \le \frac{3}{2}$, we bound $n - \gamma - z = 2 - \gamma - z \ge \frac{1}{4}$ in (5.134) (recall that $\gamma < \frac{1}{4}$ by assumption), getting

$$(5.134) \le \frac{C_\alpha}{\alpha}\,(1-y)^\alpha\,\gamma^\alpha\,4^\alpha \int_1^\infty \frac{1}{(z-y)^{1+\alpha}}\,\mathrm{d}z = \frac{C_\alpha}{\alpha^2}\,4^\alpha\,\gamma^\alpha < \infty\,.$$

Finally, we consider the contribution to $p_n(\gamma, x, y)$ of the event $\{\mathtt{s}_2 > n-\gamma\}$, i.e. by (5.133)

$$\int_{n-\gamma}^{n} \alpha \, \frac{(1-y)^\alpha}{(z-y)^{1+\alpha}} \mathrm{d}z \le \alpha \, \frac{\gamma}{(n-\frac{3}{2})^{1+\alpha}} \,, \qquad \forall n \ge 2 \,,$$

because for $y \le 1$ we have $z - y \ge n - \gamma - 1 \ge n - \frac{3}{2}$ (recall that $\gamma < \frac{1}{4}$). Recalling that $\alpha < 1$, this matches with (5.131), completing the proof of (5.84).

Next we turn to (5.83). Disintegrating over the value of $\mathtt{J}_2$, for $0 \le x \le y \le 1$ we write

$$\mathrm{P}_x\,(\mathtt{t}_2 \in [\mathtt{J}_2 - \gamma, \mathtt{J}_2] \,|\, \mathtt{t}_1 = y) = \sum_{n=2}^{\infty} \mathrm{P}_x\,(\mathtt{t}_2 \in [n-\gamma, n],\, \mathtt{J}_2 = n \,|\, \mathtt{t}_1 = y) =: \sum_{n=2}^{\infty} q_n(\gamma, x, y) \,.$$

It suffices to prove that there exists $c_\alpha \in (0,\infty)$ such that

$$q_n(\gamma, x, y) \le \frac{c_\alpha}{n^{1+\alpha}}\, \gamma^{1-\alpha} \,, \qquad \forall n \ge 2 \,,\ \forall (x,y) \in [0,1]^2_\le \,. \tag{5.136}$$

By (5.132) we can write

$$q_n(\gamma, x, y) = C_\alpha\,(1-y)^\alpha \int_{n-\gamma}^{n} \left(\int_{n-1}^{w} \frac{1}{(z-y)^{1+\alpha}\,(w-z)^{1-\alpha}} \,\mathrm{d}z \right) \frac{1}{(n-w)^\alpha} \,\mathrm{d}w \,. \tag{5.137}$$

If $n \ge 3$ then $z - y \ge n-2$ (since $y \le 1$), which plugged into in the inner integral yields

$$q_n(\gamma, x, y) \le C_\alpha \, \frac{(1-y)^\alpha}{(n-2)^{1+\alpha}} \, \frac{1}{\alpha} \int_{n-\gamma}^{n} \frac{1}{(n-w)^\alpha} \,\mathrm{d}w \le C_\alpha \, \frac{1}{(n-2)^{1+\alpha}} \, \frac{1}{\alpha} \, \frac{\gamma^{1-\alpha}}{(1-\alpha)} \,, \tag{5.138}$$

which matches with (5.136), since $n - 2 \ge n/3$ for $n \ge 3$. An analogous estimate applies also for $n = 2$, if we restrict the inner integral in (5.137) to $z \ge n - 1 + \frac{1}{2} = \frac{3}{2}$, in which case (5.138) holds with $(n-2)$ replaced by $1/2$. On the other hand, always for $n = 2$, in the range $1 \le z \le \frac{3}{2}$ we can bound $w - z \ge (2-\gamma) - \frac{3}{2} \ge \frac{1}{4}$ in the inner integral in (5.137) (recall that $\gamma < \frac{1}{4}$), getting the upper bound

$$\frac{C_\alpha\,(1-y)^\alpha}{(\frac{1}{4})^{1-\alpha}} \left(\int_1^\infty \frac{1}{(z-y)^{1+\alpha}} \,\mathrm{d}z \right) \left(\int_{2-\gamma}^{2} \frac{1}{(2-w)^\alpha} \,\mathrm{d}w \right) = \frac{4^{1-\alpha}\, C_\alpha}{\alpha(1-\alpha)} \, \gamma^{1-\alpha} \,.$$

This completes the proof of (5.136), hence of Lemma 5.20. □

5.A.2 Proof of Lemma 5.26

Recall the definition (5.104) of $\Lambda_{N,\varepsilon}$. Note that

$$\mathrm{E}[f_{N,\varepsilon}(\mathtt{s}_k, \mathtt{t}_k) \,|\, \mathtt{s}_{k-1}, \mathtt{t}_{k-1}] = \mathrm{E}_x[f_{N,\varepsilon}(\mathtt{s}_2, \mathtt{t}_2) \,|\, \mathtt{t}_1 = y]\big|_{(x,y) = (\mathtt{s}_{k-1}, \mathtt{t}_{k-1})} \,,$$

where we recall that E_x denotes expectation with respect to the regenerative set started at x, and $\mathtt{t}_1$ under P_x denotes the last visited point of $\boldsymbol{\tau}^\alpha$ in the block $[n, n+1)$, where $n = \lfloor x \rfloor$, while $\mathtt{s}_2, \mathtt{t}_2$ denote the first and last points of $\boldsymbol{\tau}^\alpha$ in the next visited block, cf. (5.79). Then we can rewrite (5.104) as

$$\Lambda_{N,\varepsilon} = \sup_{n \in \mathbb{N}_0} \ \sup_{(x,y) \in [n,n+1)^2_\le} \mathrm{E}_x \left[f_{N,\varepsilon}(\mathtt{s}_2, \mathtt{t}_2) \,\middle|\, \mathtt{t}_1 = y \right] . \tag{5.139}$$

We first note that one can set $n = 0$ in (5.139), by translation invariance, because $f_{N,\varepsilon}(s+n, t+n) = f_{N,\varepsilon}(s,t)$, cf. (5.102), and the joint law of $(\mathbf{Z}^{W,\mathtt{c}}_{\hat\beta,\hat h}(s,t), Z^{\omega,\mathtt{c}}_{\beta_N, h'_N}(Ns, Nt))_{(s,t)\in[m,m+1)^2_\le}$ does not depend on

$m \in \mathbb{N}$, by the choice of the coupling, cf. §5.5.5. Setting $n = 0$ in (5.139), we obtain

$$\Lambda_{N,\varepsilon} = \sup_{(x,y)\in[0,1)^2_\leq} \mathrm{E}_x\left(\mathbb{E}\left[\left(\frac{\mathbf{Z}^{W,\mathtt{c}}_{\hat\beta,\hat h}(\mathtt{s}_2,\mathtt{t}_2)}{\mathrm{Z}^{\omega,\mathtt{c}}_{\beta_N,h'_N}(N\mathtt{s}_2,N\mathtt{t}_2)^{1-\varepsilon}}\right)^{\frac{1}{\varepsilon}}\right]\,\middle|\, \mathtt{t}_1 = y\right). \tag{5.140}$$

In the sequel we fix $\hat\beta > 0$ and $\hat h, \hat h' \in \mathbb{R}$ with $\hat h' > \hat h$ (thus $h'_N > h_N$). Our goal is to prove that

$$\limsup_{\varepsilon\to 0}\ \limsup_{N\to\infty}\ \Lambda_{N,\varepsilon} = 0\,. \tag{5.141}$$

By Proposition 5.8, there exists a constant $C < \infty$ such that

$$\sup_{N\in\mathbb{N}}\ \sup_{0\le s\le t<\infty:\ |t-s|<1} \mathbb{E}\left[\mathrm{Z}^{\omega,\mathtt{c}}_{\beta_N,h'_N}(Ns,Nt)^2\right] = \sup_{N\in\mathbb{N}}\ \sup_{(s,t)\in[0,1]^2_\leq} \mathbb{E}\left[\mathrm{Z}^{\omega,\mathtt{c}}_{\beta_N,h'_N}(Ns,Nt)^2\right] \le C\,,$$

where the first equality holds because the law of $\mathrm{Z}^{\omega,\mathtt{c}}_{\beta_N,h'_N}(Ns,Nt)$ only depends on $t-s$. If we set

$$W_N(s,t) := \frac{\mathbf{Z}^{W,\mathtt{c}}_{\hat\beta,\hat h}(s,t)}{\mathrm{Z}^{\omega,\mathtt{c}}_{\beta_N,h'_N}(Ns,Nt)}, \qquad \mathbf{W}(s,t) := \frac{\mathbf{Z}^{W,\mathtt{c}}_{\hat\beta,\hat h}(s,t)}{\mathbf{Z}^{W,\mathtt{c}}_{\hat\beta,\hat h'}(s,t)}, \tag{5.142}$$

we can get rid of the exponent $1-\varepsilon$ in the denominator of (5.140), by Cauchy-Schwarz:

$$\mathbb{E}\left[\left(\frac{\mathbf{Z}^{W,\mathtt{c}}_{\hat\beta,\hat h}(s,t)}{\mathrm{Z}^{\omega,\mathtt{c}}_{\beta_N,h'_N}(Ns,Nt)^{1-\varepsilon}}\right)^{\frac{1}{\varepsilon}}\right] = \mathbb{E}\left[\mathrm{Z}^{\omega,\mathtt{c}}_{\beta_N,h'_N}(Ns,Nt)\, W_N(s,t)^{\frac{1}{\varepsilon}}\right] \le C^{\frac12}\,\mathbb{E}\left[W_N(s,t)^{\frac{2}{\varepsilon}}\right]^{\frac12}.$$

We can then conclude by Jensen's inequality that

$$(\Lambda_{N,\varepsilon})^2 \le C \sup_{(x,y)\in[0,1)^2_\leq} \mathrm{E}_x\left(\mathbb{E}\left[W_N(\mathtt{s}_2,\mathtt{t}_2)^{\frac{2}{\varepsilon}}\right]\,\middle|\, \mathtt{t}_{M-1} = y\right), \tag{5.143}$$

and we can naturally split the proof of our goal (5.141) in two parts:

$$\forall \varepsilon > 0: \qquad \limsup_{N\to\infty} (\Lambda_{N,\varepsilon})^2 \le C \sup_{(x,y)\in[0,1)^2_\leq} \mathrm{E}_x\left(\mathbb{E}\left[\mathbf{W}(\mathtt{s}_2,\mathtt{t}_2)^{\frac{2}{\varepsilon}}\right]\,\middle|\, \mathtt{t}_{M-1} = y\right), \tag{5.144}$$

$$\limsup_{\varepsilon\to 0}\left(\sup_{(x,y)\in[0,1)^2_\leq} \mathrm{E}_x\left[\mathbb{E}\left(\mathbf{W}(\mathtt{s}_2,\mathtt{t}_2)^{\frac{2}{\varepsilon}}\right)\,\middle|\, \mathtt{t}_1 = y\right]\right) = 0. \tag{5.145}$$

We start proving (5.144). Let $\varepsilon > 0$ be fixed. It suffices to show that the right hand side of (5.143) converges to the right hand side of (5.144) as $N \to \infty$. Writing the right hand sides of (5.143) and (5.144) respectively as $C \sup_{(x,y)\in[0,1)^2_\leq} g_N(x,y)$ and $C \sup_{(x,y)\in[0,1)^2_\leq} \mathbf{g}(x,y)$, it suffices to show that $\sup_{(x,y)\in[0,1]^2_\leq} |g_N(x,y) - \mathbf{g}(x,y)| \to 0$ as $N \to \infty$. Note that

$$\begin{aligned}
|g_N(x,y) - \mathbf{g}(x,y)| &= \left|\mathrm{E}_x\left[\mathbb{E}\left(W_N(\mathtt{s}_2,\mathtt{t}_2)^{\frac{2}{\varepsilon}}\right)\,\middle|\, \mathtt{t}_1 = y\right] - \mathrm{E}_x\left[\mathbb{E}\left(\mathbf{W}(\mathtt{s}_2,\mathtt{t}_2)^{\frac{2}{\varepsilon}}\right)\,\middle|\, \mathtt{t}_1 = y\right]\right| \\
&\le \mathrm{E}_x\left[\mathbb{E}\left(\left|W_N(\mathtt{s}_2,\mathtt{t}_2)^{\frac{2}{\varepsilon}} - \mathbf{W}(\mathtt{s}_2,\mathtt{t}_2)^{\frac{2}{\varepsilon}}\right|\right)\,\middle|\, \mathtt{t}_1 = y\right] \\
&\le \sup_{n\in\mathbb{N}_0}\ \sup_{(s,t)\in[n,n+1]^2_\leq} \mathbb{E}\left(\left|W_N(s,t)^{\frac{2}{\varepsilon}} - \mathbf{W}(s,t)^{\frac{2}{\varepsilon}}\right|\right),
\end{aligned} \tag{5.146}$$

where the last inequality holds because $n \le \mathtt{s}_2 \le \mathtt{t}_2 \le n+1$ for some integer $n \in \mathbb{N}$. The joint law of $(W_N(s,t), \mathbf{W}(s,t))_{(s,t)\in[n,n+1]^2_\le}$ does not depend on $n \in \mathbb{N}$, by our definition of the coupling in §5.5.5, hence the $\sup_{n\in\mathbb{N}_0}$ in the last line of (5.146) can be dropped, setting $n = 0$. The proof of (5.144) is thus reduced to showing that

$$\forall \varepsilon > 0: \qquad \lim_{N\to\infty} \mathbb{E}[\mathsf{S}_N] = 0, \qquad \text{with} \quad \mathsf{S}_N := \sup_{(s,t)\in[0,1]^2_\le} \left| W_N(s,t)^{\frac{2}{\varepsilon}} - \mathbf{W}(s,t)^{\frac{2}{\varepsilon}} \right|. \tag{5.147}$$

Recall the definition (5.142) of W_N and $\mathbf{W}_N$ and observe that $\lim_{N\to\infty} \mathsf{S}_N = 0$ a.s., because by construction $\mathrm{Z}^{\omega,\mathsf{c}}_{\beta_N,h_N}(Ns,Nt)$ converges a.s. to $\mathbf{Z}^{W,\mathsf{c}}_{\hat\beta,\hat h}(s,t)$, uniformly in $(s,t) \in [0,1]^2_\le$, and $\mathbf{Z}^{W,\mathsf{c}}_{\hat\beta,\hat h}(s,t) > 0$ uniformly in $(s,t) \in [0,1]^2_\le$, by [24, Theorem 2.4]. To prove that $\lim_{N\to\infty} \mathbb{E}[\,\mathsf{S}_N\,] = 0$ it then suffices to show that $(\mathsf{S}_N)_{N\in\mathbb{N}}$ is bounded in L^2 (hence uniformly integrable). To this purpose we observe

$$\mathsf{S}_N^2 \le 2 \sup_{(s,t)\in[0,1]^2_\le} W_N(s,t)^{\frac{4}{\varepsilon}} + 2 \sup_{(s,t)\in[0,1]^2_\le} \mathbf{W}(s,t)^{\frac{4}{\varepsilon}},$$

and note that $\mathbf{W}(s,t) \le 1$, because $\hat h \mapsto \mathbf{Z}^{W,\mathsf{c}}_{\hat\beta,\hat h}(s,t)$ is increasing, cf. Proposition 5.7. Finally, the first term has bounded expectation, by Proposition 5.8 and Corollary 5.9: recalling (5.142),

$$\sup_{N\in\mathbb{N}} \mathbb{E}\left[\sup_{(u,v)\in[0,1]^2_\le} W_N(s,t)^{\frac{4}{\varepsilon}} \right] \le \mathbb{E}\left[\sup_{(u,v)\in[0,1]^2_\le} \mathbf{Z}^{W,\mathsf{c}}_{\hat\beta,\hat h}(u,v)^{\frac{8}{\varepsilon}} \right]^{\frac12} \sup_{N\in\mathbb{N}} \mathbb{E}\left[\sup_{(s,t)\in[0,1]^2_\le} \mathrm{Z}^{\omega,\mathsf{c}}_{\beta_N,h'_N}(s,t)^{-\frac{8}{\varepsilon}} \right]^{\frac12} < \infty.$$

Having completed the proof of (5.144), we focus on (5.145). Let us fix $\gamma > 0$. In analogy with (5.146), we can bound the contribution to (5.145) of the event $\{\mathtt{t}_2 - \mathtt{s}_2 \ge \gamma\}$ by

$$\sup_{n\in\mathbb{N}_0} \sup_{\substack{(s,t)\in[n,n+1]^2_\le \\ |t-s|\ge\gamma}} \mathbb{E}\left[\mathbf{W}(s,t)^{\frac{2}{\varepsilon}}\right] = \sup_{\substack{(s,t)\in[0,1]^2_\le \\ |t-s|\ge\gamma}} \mathbb{E}\left[\mathbf{W}(s,t)^{\frac{2}{\varepsilon}}\right] \le \mathbb{E}\left[\sup_{\substack{(s,t)\in[0,1]^2_\le \\ |t-s|\ge\gamma}} \mathbf{W}(s,t)^{\frac{2}{\varepsilon}} \right], \tag{5.148}$$

where the equality holds because the law of $(\mathbf{W}(s,t))_{(s,t)\in[n,n+1]^2_\le}$ does not depend on $n \in \mathbb{N}_0$. Recall that by Proposition 5.7 one has, a.s., $\mathbf{W}(s,t) \le 1$ for all $(s,t) \in (0,1]^2_\le$, with $\mathbf{W}(s,t) < 1$ for $s < t$. By continuity of $(s,t) \mapsto \mathbf{W}(s,t)$ it follows that also $\sup_{(s,t)\in[0,1]^2_\le:\ |t-s|\ge\gamma} \mathbf{W}(s,t) < 1$, a.s., hence the right hand side of (5.148) vanishes as $\varepsilon \to 0$, for any fixed $\gamma > 0$, by dominated convergence. This means that in order to prove (5.145) we can focus on the event $\{\mathtt{t}_2 - \mathtt{s}_2 < \gamma\}$, and note that

$$\sup_{(x,y)\in[0,1)^2_\le} \mathrm{E}_x\left[\mathbb{E}\left(\mathbf{W}(\mathtt{s}_2,\mathtt{t}_2)^{\frac{2}{\varepsilon}}\right) \mathbb{1}_{\{\mathtt{t}_2-\mathtt{s}_2<\gamma\}} \,\middle|\, \mathtt{t}_1 = y \right] \le \sup_{(x,y)\in[0,1)^2_\le} \mathrm{P}_x(\mathtt{t}_2 - \mathtt{s}_2 \le \gamma \,|\, \mathtt{t}_1 = y),$$

because $\mathbf{W}(s,t) \le 1$. Since $\gamma > 0$ was arbitrary, in order to prove (5.145) it is enough to show that

$$\lim_{\gamma\to0} \sup_{(x,y)\in[0,1)^2_\le} \mathrm{P}_x(\mathtt{t}_2 - \mathtt{s}_2 \le \gamma \,|\, \mathtt{t}_1 = y) = 0. \tag{5.149}$$

This is a consequence of relation (5.84) in Lemma 5.20, which concludes the proof of Lemma 5.26. □

5.A.3 Proof of Lemma 5.27

We omit the proof of relation (5.118), because it is analogous to (and simpler than) the proof of relation (5.106) in Lemma 5.26: compare the definition of $f_{N,\varepsilon}$ in (5.102) with that of $g_{N,\varepsilon}$ in (5.114), and the definition of $\Lambda_{N,\varepsilon}$ in (5.104) with that of $\Lambda_{N,\varepsilon,p}$ in (5.117) (note that the exponent p in (5.117) can be brought inside the $\mathbb{E}$-expectation in (5.114), by Jensen's inequality).

In order complete the proof of Lemma 5.27, we state an auxiliary Lemma, proved in §5.A.4 below. Recall that $\mathrm{R}_t(M,(x_k,y_k)_{k=1}^M)$ was defined in (5.110), for $t, M \in \mathbb{N}$ and $x_k, y_k \in \frac{1}{N}\mathbb{N}_0$ satisfying the constraints $0 = x_1 \le y_1 < x_2 \le y_2 < \ldots < x_M \le y_M \le t$. Also recall that $L : \mathbb{N} \to (0,\infty)$ denotes the slowly varying function appearing in (5.1), and we set $L(0) = 1$ for convenience.

Lemma 5.29. *Relation* (5.115) *holds for suitable functions r_N, $\tilde{r}_N$, satisfying the following relations:*

- *there is $C \in (0,\infty)$ such that for all $N \in \mathbb{N}$ and all admissible y', x, y, resp. z, t,*

$$r_N(y',x,y) \le C\, \frac{L(N(x-y'))}{L(N(\lceil y'\rceil - y'))}\, \frac{L(N(\lceil y\rceil - y))}{L(N(y-x))}\,, \qquad \tilde{r}_N(z,t) \le C\, \frac{L(N(t-z))}{L(N(\lceil z\rceil - z))}\,; \tag{5.150}$$

- *for all $\eta > 0$ there is $M_0 = M_0(\eta) < \infty$ such that for all $N \in \mathbb{N}$ and for admissible y', x, y*

$$r_N(0,0,y) \le (1+\eta)\, \frac{L(N(\lceil y\rceil - y))}{L(Ny)}\,, \qquad \text{if} \quad y \ge \frac{M_0}{N}\,; \tag{5.151}$$

$$r_N(y',x,y) \le (1+\eta)\, \frac{L(N(x-y'))}{L(N(\lceil y'\rceil - y'))}\, \frac{L(N(\lceil y\rceil - y))}{L(N(y-x))}\,, \qquad \text{if} \quad y-x \ge \frac{M_0}{N}\,, \quad x-y' \ge \frac{M_0}{N}\,. \tag{5.152}$$

We can now prove relations (5.119), (5.120). By Potter's bounds [14, Theorem 1.5.6], for any $\delta > 0$ there is a constant $c_\delta > 0$ such that $L(m)/L(\ell) \le c_\delta \max\{\frac{m+1}{\ell+1}, \frac{\ell+1}{m+1}\}^\delta$ for all $m, \ell \in \mathbb{N}_0$ (the "+1" is because we allow ℓ, m to attain the value 0). Looking at (5.150)-(5.152), recalling that the admissible values of y', x, y are such that $\lceil y'\rceil - y' \le x - y'$ and $y - x \le 1$, $\lceil y\rceil - y \le 1$, we can estimate

$$\begin{aligned}\frac{L(N(x-y'))}{L(N(\lceil y'\rceil - y'))}\, \frac{L(N(\lceil y\rceil - y))}{L(N(y-x))} &\le c_\delta^2 \left(\frac{x-y'+\frac{1}{N}}{\lceil y'\rceil - y' + \frac{1}{N}}\right)^\delta \max\left\{\frac{y-x+\frac{1}{N}}{\lceil y\rceil - y + \frac{1}{N}}, \frac{\lceil y\rceil - y + \frac{1}{N}}{y-x+\frac{1}{N}}\right\}^\delta \\ &\le 2^\delta c_\delta^2 \left(\frac{x-y'+\frac{1}{N}}{\lceil y'\rceil - y' + \frac{1}{N}}\right)^\delta \frac{1}{(\lceil y\rceil - y + \frac{1}{N})^\delta}\, \frac{1}{(y-x+\frac{1}{N})^\delta}\,.\end{aligned}$$

We now plug in $y' = \mathtt{s}_{k-1}^{(N)}$, $x = \mathtt{s}_k^{(N)}$, $y = \mathtt{t}_k^{(N)}$ (so that $\lceil y'\rceil = \mathtt{J}_{k-1}$ and $\lceil y\rceil = \mathtt{J}_k$). The first relation in (5.150) then yields

$$\begin{aligned} r_N\left(\mathtt{t}_{k-1}^{(N)}, \mathtt{s}_k^{(N)}, \mathtt{t}_k^{(N)}\right) &\le C\, 2^\delta c_\delta^2 \left(\frac{\mathtt{s}_k^{(N)} - \mathtt{t}_{k-1}^{(N)} + \frac{1}{N}}{\mathtt{J}_{k-1} - \mathtt{t}_{k-1}^{(N)} + \frac{1}{N}}\right)^\delta \frac{1}{(\mathtt{J}_k - \mathtt{t}_k^{(N)} + \frac{1}{N})^\delta}\, \frac{1}{(\mathtt{t}_k^{(N)} - \mathtt{s}_k^{(N)} + \frac{1}{N})^\delta} \\ &\le C\, 2^\delta c_\delta^2 \left(\frac{\mathtt{s}_k - \mathtt{t}_{k-1}}{\mathtt{J}_{k-1} - \mathtt{t}_{k-1}}\right)^\delta \frac{1}{(\mathtt{J}_k - \mathtt{t}_k)^\delta}\, \frac{1}{(\mathtt{t}_k - \mathtt{s}_k)^\delta}\,,\end{aligned}$$

where the last inequality holds by monotonicity, since $\mathtt{s}_k^{(N)} \le \mathtt{s}_k$, $\mathtt{t}_i^{(N)} \le \mathtt{t}_i$ for $i = k-1, k$ and $\mathtt{t}_k^{(N)} - \mathtt{s}_k^{(N)} + \frac{1}{N} \ge \mathtt{t}_k - \mathtt{s}_k$ by definition (5.108). Setting $C'_\delta := C\, 2^\delta c_\delta^2$, by the regenerative property

$$\mathrm{E}\left[r_N\left(\mathtt{t}_{k-1}^{(N)}, \mathtt{s}_k^{(N)}, \mathtt{t}_k^{(N)}\right)^{\frac{p}{\varepsilon}} \,\middle|\, \mathtt{s}_{k-1}\, \mathtt{t}_{k-1}\right] \le (C'_\delta)^{\frac{p}{\varepsilon}}\, \mathrm{E}_x\left[\left(\frac{\mathtt{s}_2 - y}{1-y}\right)^{\frac{\delta p}{\varepsilon}} \frac{1}{(\mathtt{J}_2 - \mathtt{t}_2)^{\frac{\delta p}{\varepsilon}}}\, \frac{1}{(\mathtt{t}_2 - \mathtt{s}_2)^{\frac{\delta p}{\varepsilon}}} \,\middle|\, \mathtt{t}_1 = y\right],$$

with $(x,y) = (\mathtt{s}_{k-1}, \mathtt{t}_{k-1})$. Since $\mathrm{E}[XYZ] \le (\mathrm{E}[X^3]\mathrm{E}[Y^3]\mathrm{E}[Z^3])^{1/3}$ by Hölder's inequality, we split the expected value in the right hand side in three parts, estimating each term separately.

First, given $x, y \in [n, n+1)$ for some $n \in \mathbb{N}$, then $\mathtt{t}_1 = \mathtt{g}_n(\tau^\alpha)$ and $\mathtt{s}_2 = \mathtt{d}_n(\tau^\alpha)$, hence by (5.78)

$$\mathrm{E}_x\left[\left(\frac{\mathtt{s}_2 - y}{1-y}\right)^{\frac{3\delta p}{\varepsilon}} \,\middle|\, \mathtt{t}_1 = y\right] = \mathrm{E}_x\left[\left(\frac{\mathtt{d}_n(\tau^\alpha) - y}{1-y}\right)^{\frac{3\delta p}{\varepsilon}} \,\middle|\, \mathtt{g}_n(\tau^\alpha) = y\right] = \int_n^\infty \left(\frac{v-y}{n-y}\right)^{\frac{3\delta p}{\varepsilon}} \frac{(n-y)^\alpha}{(v-y)^{1+\alpha}}\, \mathrm{d}v\,,$$

and the change of variable $z := \frac{v-y}{n-y}$ yields

$$\mathrm{E}_x\left[\left(\frac{\mathtt{s}_2 - y}{1-y}\right)^{\frac{3\delta p}{\varepsilon}} \,\middle|\, \mathtt{t}_1 = y\right] = \int_1^\infty z^{\frac{3\delta p}{\varepsilon}-1-\alpha}\,\mathrm{d}z = \frac{1}{\alpha - \frac{3\delta p}{\varepsilon}} =: C_1 < \infty\,, \qquad \text{if} \quad \delta < \frac{\alpha\varepsilon}{3p}\,. \tag{5.153}$$

Next, since $\mathrm{E}[X^{-a}] = \int_0^\infty \mathrm{P}(X^{-a} \ge t)\,\mathrm{d}t = \int_0^\infty \mathrm{P}(X \le t^{-1/a})\,\mathrm{d}t$ for any random variable $X \ge 0$,

$$\begin{aligned}\mathrm{E}_x\left[\frac{1}{(\mathtt{J}_2 - \mathtt{t}_2)^{\frac{3\delta p}{\varepsilon}}} \,\middle|\, \mathtt{t}_1 = y\right] &= \int_0^\infty \mathrm{P}_x\left(\mathtt{J}_2 - \mathtt{t}_2 \le \gamma^{-\frac{\varepsilon}{3\delta p}} \,\middle|\, \mathtt{t}_1 = y\right)\mathrm{d}\gamma \\ &\le A_\alpha \int_0^\infty \min\{1, \gamma^{-(1-\alpha)\frac{\varepsilon}{3\delta p}}\}\,\mathrm{d}\gamma =: C_2 < \infty\,, \qquad \text{if} \quad \delta < \frac{(1-\alpha)\varepsilon}{3p}\,,\end{aligned} \tag{5.154}$$

having used (5.83). Analogously, using (5.84),

$$\mathrm{E}_x\left[\frac{1}{(\mathtt{t}_2 - \mathtt{s}_2)^{\frac{3\delta p}{\varepsilon}}} \,\middle|\, \mathtt{t}_1 = y\right] \le B_\alpha \int_0^\infty \min\{1, \gamma^{-\alpha\frac{\varepsilon}{3\delta p}}\}\,\mathrm{d}\gamma =: C_3 < \infty\,, \qquad \text{if} \quad \delta < \frac{\alpha\varepsilon}{3p}\,. \tag{5.155}$$

In conclusion, given $\varepsilon \in (0,1)$ and $p \ge 1$, if we fix $\delta < \min\{\alpha, 1-\alpha\}\frac{\varepsilon}{3p}$, by (5.153)-(5.154)-(5.155) there are constants $C_1, C_2, C_3 < \infty$ (depending on ε, p) such that for all $N \in \mathbb{N}$ and $k \ge 2$

$$\mathrm{E}\left[r_N\left(\mathtt{t}_{k-1}^{(N)}, \mathtt{s}_k^{(N)}, \mathtt{t}_k^{(N)}\right)^{\frac{p}{\varepsilon}} \,\middle|\, \mathtt{s}_{k-1}\,\mathtt{t}_{k-1}\right] \le (C'_\delta)^{\frac{p}{\varepsilon}}\,(C_1\,C_2\,C_3)^{1/3} =: C_{\varepsilon,p} < \infty\,, \tag{5.156}$$

which proves (5.119). Relation (5.120) is proved with analogous (and simpler) estimates, using the second relation in (5.150).

Finally, we prove relations (5.121)-(5.122), exploiting the upper bound (5.152) in which we plug $y' = \mathtt{s}_{k-1}^{(N)}$, $x = \mathtt{s}_k^{(N)}$, $y = \mathtt{t}_k^{(N)}$ (recall that $\lceil y' \rceil = \mathtt{J}_{k-1}$ and $\lceil y \rceil = \mathtt{J}_k$). We recall that, by the uniform convergence theorem of slowly varying functions [14, Theorem 1.2.1], $\lim_{N\to\infty} L(Na)/L(Nb) = 1$ *uniformly for* a, b *in a compact subset of* $(0,\infty)$. It follows by (5.152) that for all $\eta > 0$ and for all $\gamma, \tilde\gamma \in (0,1)$, $T \in (0,\infty)$ there is $\hat N_0 = \hat N_0(\gamma, \tilde\gamma, \eta, T) < \infty$ such that for all $N \ge \hat N_0$ and for $k \ge 2$

$$r_N\left(\mathtt{t}_{k-1}^{(N)}, \mathtt{s}_k^{(N)}, \mathtt{t}_k^{(N)}\right) \le (1+\eta)^2$$
$$\text{on the event} \quad \{\mathtt{J}_{k-1} - \mathtt{t}_{k-1} \ge \gamma\} \cap \{\mathtt{J}_k - \mathtt{t}_k \ge \tilde\gamma\,,\ \mathtt{t}_k - \mathtt{s}_k \ge \tilde\gamma\,,\ \mathtt{s}_k - \mathtt{t}_{k-1} \le T\}\,.$$

Consequently, on the event $\{\mathtt{J}_{k-1} - \mathtt{t}_{k-1} \ge \gamma\} = \{\mathtt{t}_{k-1} \le \mathtt{J}_{k-1} - \gamma\}$ we can write

$$\begin{aligned}&\mathrm{E}\left[r_N\left(\mathtt{t}_{k-1}^{(N)}, \mathtt{s}_k^{(N)}, \mathtt{t}_k^{(N)}\right)^{\frac{p}{\varepsilon}} \,\middle|\, \mathtt{s}_{k-1}\,\mathtt{t}_{k-1}\right] \\ &\quad\le (1+\eta)^{\frac{2p}{\varepsilon}} + \mathrm{E}\left[r_N\left(\mathtt{t}_{k-1}^{(N)}, \mathtt{s}_k^{(N)}, \mathtt{t}_k^{(N)}\right)^{\frac{p}{\varepsilon}}\,\mathbb{1}_{\{\mathtt{J}_k - \mathtt{t}_k \ge \tilde\gamma,\, \mathtt{t}_k - \mathtt{s}_k \ge \tilde\gamma,\, \mathtt{s}_k - \mathtt{t}_{k-1} \le T\}^c} \,\middle|\, \mathtt{s}_{k-1}\,\mathtt{t}_{k-1}\right] \\ &\quad\le (1+\eta)^{\frac{2p}{\varepsilon}} + \sqrt{C_{\varepsilon,2p}\,\mathrm{P}_x(\{\mathtt{J}_2 - \mathtt{t}_2 \ge \tilde\gamma,\, \mathtt{t}_2 - \mathtt{s}_2 \ge \tilde\gamma,\, \mathtt{s}_2 - y \le T\}^c \,|\, \mathtt{t}_1 = y)}\,,\end{aligned}$$

where in the last line we have applied Cauchy-Schwarz, relation (5.156) and the regenerative property, with $(x,y) = (\mathtt{s}_{k-1}, \mathtt{t}_{k-1})$. Since for $x, y \in [n, n+1)$ one has $\mathtt{t}_1 = \mathsf{g}_n(\boldsymbol{\tau}^\alpha)$ and $\mathtt{s}_2 = \mathsf{d}_n(\boldsymbol{\tau}^\alpha)$, by (5.78)

$$\mathrm{P}_x(\mathtt{s}_2 - y > T \,|\, \mathtt{t}_1 = y) = \mathrm{P}_x(\mathsf{d}_n(\boldsymbol{\tau}^\alpha) > T + y \,|\, \mathsf{g}_n(\boldsymbol{\tau}^\alpha) = y) = \int_{y+T}^\infty \frac{(n-y)^\alpha}{(v-y)^{1+\alpha}}\,\mathrm{d}v \le \frac{1}{\alpha\,T^\alpha}\,,$$

because $n - y \le 1$. Applying relations (5.83)-(5.84), we have shown that for $N \ge \hat N_0$ and $k \ge 2$, on the event $\{\mathtt{t}_{k-1} \le \mathtt{J}_{k-1} - \gamma\}$ we have the estimate

$$\mathrm{E}\left[r_N \left(\mathtt{t}_{k-1}^{(N)}, \mathtt{s}_k^{(N)}, \mathtt{t}_k^{(N)} \right)^{\frac{p}{\varepsilon}} \,\middle|\, \mathtt{s}_{k-1}\, \mathtt{t}_{k-1} \right] \le (1+\eta)^{\frac{2p}{\varepsilon}} + \sqrt{C_{\varepsilon,2p}\, (A_\alpha \tilde{\gamma}^{1-\alpha} + B_\alpha \tilde{\gamma}^{\alpha} + \alpha^{-1}\, T^{-\alpha})}\,. \tag{5.157}$$

We can finally fix η, $\tilde{\gamma}$ small enough and T large enough (depending only on ε and p) so that the right hand side of (5.157) is less than 2. This proves relation (5.121), for all $\varepsilon \in (0,1)$, $p \ge 1$, $\gamma \in (0,1)$, with $\tilde{N}_0(\varepsilon, p, \gamma) := \hat{N}_0(\gamma, \tilde{\gamma}, \eta, T)$. Relation (5.122) is proved similarly, using (5.151). □

5.A.4 Proof of Lemma 5.29

We recall that the random variables $\mathtt{s}_k^{(N)}, \mathtt{t}_k^{(N)}, \mathtt{m}_t^{(N)}$ in the numerator of (5.110) refer to the rescaled renewal process τ/N, cf. Definition 5.18. By (5.1)-(5.8), we can write the numerator in (5.110), which we call $\mathtt{L}_M$, as follows: for $0 = x_1 \le y_1 < x_2 \le y_2 < \ldots < x_M \le y_M < t$, with $x_i, y_i \in \frac{1}{N}\mathbb{N}_0$,

$$\mathtt{L}_M = u(Ny_1) \left(\prod_{i=2}^{M} K(N(x_i - y_{i-1}))\, u(N(y_i - x_i)) \right) \bar{K}(N(t - y_M))\,, \tag{5.158}$$

where we set $\bar{K}(\ell) := \sum_{n>\ell} K(n)$. Analogously, using repeatedly (5.76) and the regenerative property, the denominator in (5.110), which we call $\mathtt{I}_M$, can be rewritten as

$$\mathtt{I}_M := \int\limits_{\substack{u_i \in [x_i, x_i + \frac{1}{N}],\, 2 \le i \le M \\ v_i \in [y_i, y_i + \frac{1}{N}],\, 1 \le i \le M}} \!\!\cdots\!\! \int \frac{C_\alpha}{v_1^{1-\alpha}} \left(\prod_{i=2}^{M} \frac{C_\alpha\, \mathbb{1}_{\{u_i < v_i\}}}{(u_i - v_{i-1})^{1+\alpha}\, (v_i - u_i)^{1-\alpha}} \right) \frac{1}{\alpha\, (t - v_M)^\alpha}\, \mathrm{d}v_1\, \mathrm{d}u_2\, \mathrm{d}v_2 \cdots \mathrm{d}u_M\, \mathrm{d}v_M\,. \tag{5.159}$$

Bounding uniformly

$$u_i - v_{i-1} \le x_i - y_{i-1} + \tfrac{1}{N}, \qquad v_i - u_i \le y_i - x_i + \tfrac{1}{N}, \qquad t - v_M \le t - y_M + \tfrac{1}{N}\,, \tag{5.160}$$

we obtain a lower bound for $\mathtt{I}_M$ which is factorized as a product over blocks:

$$\begin{aligned} &\frac{1}{N^{2M-1}}\, \frac{C_\alpha}{(x_1 + \frac{1}{N})^{1-\alpha}} \left(\prod_{i=2}^{M} \frac{C_\alpha}{(x_i - y_{i-1} + \frac{1}{N})^{1+\alpha}\, (y_i - x_i + \frac{1}{N})^{1-\alpha}} \right) \frac{1}{\alpha\, (t - y_M)^\alpha} \\ &\quad = \frac{C_\alpha}{(Nx_1 + 1)^{1-\alpha}} \left(\prod_{i=2}^{M} \frac{C_\alpha}{(N(x_i - y_{i-1}) + 1)^{1+\alpha}\, (N(y_i - x_i) + 1)^{1-\alpha}} \right) \frac{1}{\alpha\, (N(t - y_M) + 1)^\alpha}\,. \end{aligned} \tag{5.161}$$

Looking back at (5.158) and recalling (5.110), it follows that relation (5.115) holds with

$$\begin{aligned} r_N(y', x, y) &:= (N(x - y') + 1)^{1+\alpha}\, K(N(x - y'))\, \frac{(N(y - x) + 1)^{1-\alpha}}{C_\alpha}\, u(N(y - x)) \left\{ \frac{L(N(\lceil y \rceil - y))}{L(N(\lceil y' \rceil - y'))} \right\}, \\ \tilde{r}_N(z, t) &:= \alpha\, (N(t - z) + 1)^\alpha\, \frac{\bar{K}(N(t - z))}{L(N(\lceil z \rceil - z))}\,, \end{aligned}$$

where we have "artificially" added the last terms inside the brackets, which get simplified telescopically when one considers the product in (5.115). (In order to define $r_N(y', x, y)$ also when $y' = x = 0$, which is necessary for the first term in the product in (5.115), we agree that $K(0) := 1$.)

Recalling (5.1) and (5.8), there is some constant $\mathsf{C} \in (1, \infty)$ such that for all $n \in \mathbb{N}_0$

$$K(n) \le \mathsf{C} \frac{L(n)}{(n+1)^{1+\alpha}}\,, \qquad \bar{K}(n) \le \mathsf{C} \frac{L(n)}{\alpha(n+1)^\alpha}\,, \qquad u(n) \le \mathsf{C} \frac{C_\alpha}{L(n)\, (n+1)^{1+\alpha}}\,. \tag{5.162}$$

Plugging these estimates into the definitions of r_N, $\tilde{r}_N$ yields the first and second relations in (5.150), with $C = \mathtt{C}^2$ and $C = \mathtt{C}$, respectively. Finally, given $\eta > 0$ there is $M_0 = M_0(\eta) < \infty$ such that for $n \geq M_0$ one can replace $\mathtt{C}$ by $(1+\eta)$ in (5.162), which yields (5.151) and (5.152). □

Remark 5.30. To prove $f^{(1)} \prec f^{(2)}$ we have shown that it is possible to give an upper bound, cf. (5.115), for the Radon-Nikodym derivative R_t by suitable functions r_N and $\tilde{r}_N$ satisfying Lemma 5.27. Analogously, to prove the complementary step $f^{(3)} \prec f^{(2)}$, that we do not detail, one would need an analogous upper bound for the *inverse* of the Radon-Nikodym derivative, i.e.

$$\mathrm{R}_t\left(M, (x_k, y_k)_{k=1}^M\right)^{-1} \leq \left\{\prod_{\ell=1}^{M} q_N(y_{\ell-1}, x_\ell, y_\ell)\right\} \tilde{q}_N(y_M, t)\,, \tag{5.163}$$

for suitable functions q_N and $\tilde{q}_N$ that satisfy conditions similar to r_N and $\tilde{r}_N$ in Lemma 5.29, thus yielding an analogue of Lemma 5.27. We now show how to prove this.

We start to observe that the inverse of the Radon-Nikodym derivative is given by the ratio between $\mathtt{I}_M$ and $\mathtt{L}_M$ – cf. (5.159) and (5.158) respectively –, i.e., $\mathrm{R}_t^{-1} = \mathtt{I}_M/\mathtt{L}_M$. Therefore we need to show that the multiple integral $\mathtt{I}_M$ admits an upper bound given by a suitable factorization, analogous to (5.161). The simple strategy of using uniform bounds that are complementary to (5.160), i.e. $u_i - v_{i-1} \geq x_i - y_{i-1} - \frac{1}{N}$ etc., does not work when $x_i = y_{i-1}$, so some additional care is needed.

We start by "adding artificially" boundary terms in $\mathtt{I}_M$, rewriting the integrand in (5.159) as

$$\frac{C_\alpha}{v_1^{1-\alpha}\,\alpha\,(1-v_1)^\alpha}\left(\prod_{i=2}^{M}\frac{C_\alpha\,\mathbb{1}_{\{u_i<v_i\}}}{(u_i-v_{i-1})^{1+\alpha}\,(v_i-u_i)^{1-\alpha}}\,\frac{\alpha\,(\lceil y_{i-1}\rceil - v_{i-1})^\alpha}{\alpha\,(\lceil y_i\rceil - v_i)^\alpha}\right)\frac{\alpha\,(\lceil y_M\rceil - v_M)^\alpha}{\alpha\,(t-v_M)^\alpha}\,, \tag{5.164}$$

where the added terms disappear telescopically. Next we introduce the functions

$$f(n) := \begin{cases}\frac{1}{(n-1)^{1+\alpha}}, & n\geq 2\\ \frac{1}{\alpha}, & n=1\end{cases}\quad,\quad g(n) := \begin{cases}\frac{C_\alpha}{(n-1)^{1-\alpha}}, & n\geq 2\\ \frac{4}{\alpha}\,C_\alpha & n=0,\,1\end{cases}\quad,\quad \bar{f}(n) := \begin{cases}\frac{1}{\alpha(n-1)^\alpha}, & n\geq 2\\ \frac{1}{\alpha(1-\alpha)}, & n=1\end{cases} \tag{5.165}$$

and we show that they provide a factorization of $\mathtt{I}_M$. More precisely, given $y', x, y \in \frac{1}{N}\mathbb{N}_0$ such that $0 < y' < \lceil y'\rceil \leq x \leq y < \lceil y\rceil \leq t$, we are going to show that for all $v' \in [y', y' + \frac{1}{N})$

$$\int\!\!\cdots\!\!\int_{\substack{u\in[x,x+\frac{1}{N})\\ v\in[y,y+\frac{1}{N})}} \frac{C_\alpha\mathbb{1}_{\{u<v\}}}{(u-v')^{1+\alpha}(v-u)^{1-\alpha}}\,\frac{\alpha(\lceil y'\rceil - v')^\alpha}{\alpha(\lceil y\rceil - v)^\alpha}\mathrm{d}u\,\mathrm{d}v \;\leq f\Big(N(x-y')\Big)\,g\Big(N(y-x)\Big)\frac{\bar{f}\Big(N(\lceil y\rceil - y)\Big)}{\bar{f}\Big(N(\lceil y'\rceil - y') + 1\Big)}. \tag{5.166}$$

Analogously the first and the last term in (5.164) are estimated by

$$\int\!\!\cdots\!\!\int_{v\in[y,y+\frac{1}{N})} \frac{C_\alpha}{v^{1-\alpha}}\,\frac{1}{\alpha(1-v)^\alpha}\mathrm{d}u\,\mathrm{d}v \;\leq g\Big(N\,y\Big)\,\bar{f}\Big(N(1-y)\Big), \tag{5.167}$$

$$\frac{\alpha\,(\lceil y\rceil - v)^\alpha}{\alpha\,(t-v)^\alpha} \;\leq\; \frac{\bar{f}\Big(N(t-y)\Big)}{\bar{f}\Big(N(\lceil y\rceil - y) + 1\Big)}, \qquad \forall v \in [y, y+\frac{1}{N}). \tag{5.168}$$

Applying (5.168), then iteratively (5.167) and finally (5.166), we obtain

$$\begin{aligned}\mathtt{I}_M \le g\big(N y_1\big)\, \bar{f}\big(N(1-y_1)\big) &\left(\prod_{i=1}^{M} f\big(N(x_i - y_{i-1})\big)\, g\big(N(y_i - x_i)\big) \frac{\bar{f}\big(N(\lceil y_i \rceil - y_i)\big)}{\bar{f}\big(N(\lceil y_{i-1} \rceil - y_{i-1}) + 1\big)} \right) \times \\ &\times \frac{\bar{f}\big(N(t - y_M)\big)}{\bar{f}\big(N(\lceil y_M \rceil - y_M) + 1\big)}.\end{aligned} \tag{5.169}$$

Analogously we observe that $\mathtt{L}_M$ in (5.158) admits a similar (alternative) factorization, namely for $0 = x_1 \le y_1 < x_2 \le y_2 < \ldots < x_M \le y_M < t$, with $x_i, y_i \in \frac{1}{N}\mathbb{N}_0$,

$$\mathtt{L}_M = u(Ny_1)\bar{K}(N(1-y_1)) \left(\prod_{i=2}^{M} K(N(x_i - y_{i-1}))\, u(N(y_i - x_i)) \frac{\bar{K}(N(\lceil y_i \rceil - y_i))}{\bar{K}(N(\lceil y_{i-1} \rceil - y_{i-1}))} \right) \frac{\bar{K}(N(t - y_M))}{\bar{K}(N(\lceil y_M \rceil - y_M))}. \tag{5.170}$$

Summarizing, since $\mathrm{R}_t^{-1} = \mathtt{I}_M/\mathtt{L}_M$, an upper bound is provided by the ratio between (5.169) and (5.170). This leads directly to the factorization stated in (5.163), where we set

$$\begin{aligned} q_N(y', x, y) &= \frac{f\big(N(x-y')\big)\, g\big(N(y-x)\big)}{K(N(x-y'))\, u(N(y-x))} \frac{\bar{K}(N(\lceil y' \rceil - y'))}{\bar{f}\big(N(\lceil y' \rceil - y') + 1\big)} \frac{\bar{f}\big(N(\lceil y \rceil - y)\big)}{\bar{K}(N(\lceil y \rceil - y))}, \\ \tilde{q}_N(z,t) &= \frac{\bar{K}(N(\lceil z \rceil - z))}{\bar{f}\big(N(\lceil z \rceil - z) + 1\big)} \frac{\bar{f}\big(N(t-z)\big)}{\bar{K}(N(t-z))}. \end{aligned} \tag{5.171}$$

Finally, using lower bounds analogous to the upper bounds in (5.162), it is easy to check that $q_N(y', x, y)$ and $\tilde{q}_N(z,t)$ satisfy properties analogous to the ones in Lemma 5.29, except that the right hand sides therein are replaced by their inverses. This is immaterial, however, because the proof of Lemma 5.27 was based on Potter bounds, which are symmetric with respect to inversion.

To conclude, we have to prove (5.166), (5.167) and (5.168). We detail the most interesting case, namely (5.166), since the other two follow similarly (and they are actually simpler). We proceed by steps, by considering different subcases. We recall that $0 < y' < \lceil y' \rceil \le x \le y < \lceil y \rceil \le t$, where $y', x, y \in \frac{1}{N}\mathbb{N}_0$ and $v' \in [y', y' + \frac{1}{N})$.

In the first step we assume $y - x \ge \frac{2}{N}$. In such case we estimate the term $(v-u)^{-(1-\alpha)}$ in (5.166) by $(y - x - \frac{1}{N})^{-(1-\alpha)} = N^{1-\alpha}\, g(N(y-x))$, by (5.165). This allow to split the integral in two parts, each one depending on a single variable:

$$\begin{aligned} &\int_x^{x+\frac{1}{N}} \left(\int_y^{y+\frac{1}{N}} \frac{C_\alpha \mathbb{1}_{\{u<v\}}}{(u-v')^{1+\alpha}(v-u)^{1-\alpha}} \frac{\alpha(\lceil y' \rceil - v')^\alpha}{\alpha(\lceil y \rceil - v)^\alpha} \mathrm{d}v \right) \mathrm{d}u \\ &\qquad \le \left(\int_x^{x+\frac{1}{N}} \frac{\alpha(\lceil y' \rceil - v')^\alpha}{(u-v')^{1+\alpha}} \mathrm{d}u \right) N^{1-\alpha}\, g(N(y-x)) \left(\int_y^{y+\frac{1}{N}} \frac{1}{\alpha(\lceil y \rceil - v)^\alpha} \mathrm{d}v \right). \end{aligned} \tag{5.172}$$

- To estimate the first integral we consider two different cases. If $x = \lceil y' \rceil$, we replace the upper extreme $x + \frac{1}{N}$ by $+\infty$ and the integral becomes $1 = f\big(1\big)/\bar{f}\big(2\big)$. On the other hand, if $x - \lceil y' \rceil \ge \frac{1}{N}$, and thus $x - y' \geq \frac{2}{N}$, we have

$$\frac{\alpha(\lceil y'\rceil - v')^{\alpha}}{(u - v')^{1+\alpha}} \leq \frac{\alpha(\lceil y'\rceil - y')^{\alpha}}{(x - y' - \frac{1}{N})^{1+\alpha}} = N \frac{f\Big(N(x - y')\Big)}{\bar{f}\Big(N(\lceil y'\rceil - y') + 1\Big)},$$

which shows that an upper bound for the first integral is always $f(N(x - y'))/\bar{f}(N(\lceil y'\rceil - y') + 1)$, explaining two of the terms in the right hand side of (5.166).

- If $\lceil y\rceil = y + \frac{1}{N}$, the second integral equals $\alpha^{-1}(1-\alpha)^{-1}N^{-(1-\alpha)} = N^{-(1-\alpha)}\bar{f}\big(1\big)$. On the other hand, if $\lceil y\rceil - y \geq \frac{2}{N}$, then we can replace $(\lceil y\rceil - v)^{-\alpha}$ by $(\lceil y\rceil - y - \frac{1}{N})^{-\alpha}$ and the integral is estimated by $\alpha^{-1}N^{-1}(\lceil y\rceil - y - \frac{1}{N})^{-\alpha} = N^{-(1-\alpha)}\bar{f}(N(\lceil y\rceil - y))$, completing the proof of (5.166).

It remains to consider the case $y - x \leq \frac{1}{N}$ (i.e., $y = x$ or $y = x + \frac{1}{N}$). To complete the proof we consider three subcases. We also assume $N \geq 3$.

- We first assume that $\lceil y\rceil - y = \frac{1}{N}$. Then necessarily $x - y' \geq \frac{2}{N}$ (because $y - x \leq \frac{1}{N}$ and $N \geq 3$). Bounding $u - v' \geq x - y' - \frac{1}{N}$ and $(\lceil y'\rceil - v')^{\alpha} \leq (\lceil y'\rceil - y')^{\alpha}$ and replacing $\int_x^{x+\frac{1}{N}}$ with $\int_{y-\frac{1}{N}}^{y+\frac{1}{N}}$, we get an upper bound, i.e.,

$$\int_x^{x+\frac{1}{N}}\left(\int_y^{y+\frac{1}{N}} \frac{C_\alpha \mathbb{1}_{\{u<v\}}}{(u-v')^{1+\alpha}(v-u)^{1-\alpha}} \frac{\alpha(\lceil y'\rceil - v')^{\alpha}}{\alpha(\lceil y\rceil - v)^{\alpha}} \mathrm{d}v\right)\mathrm{d}u$$
$$\leq N \frac{f\Big(N(x-y')\Big)}{\bar{f}\Big(N(\lceil y'\rceil - y') + 1\Big)} \int_{\lceil y\rceil - \frac{2}{N}}^{\lceil y\rceil}\left(\int_{\lceil y\rceil - \frac{1}{N}}^{\lceil y\rceil} \frac{C_\alpha \mathbb{1}_{\{u<v\}}}{(v-u)^{1-\alpha}(\lceil y\rceil - v)^{\alpha}}\mathrm{d}v\right)\mathrm{d}u\,.$$

By direct computation

$$\int_{\lceil y\rceil - \frac{2}{N}}^{\lceil y\rceil}\left(\int_{\lceil y\rceil - \frac{1}{N}}^{\lceil y\rceil} \frac{C_\alpha \mathbb{1}_{\{u<v\}}}{(v-u)^{1-\alpha}(\lceil y\rceil - v)^{\alpha}}\mathrm{d}v\right)\mathrm{d}u = \int_{\lceil y\rceil - \frac{1}{N}}^{\lceil y\rceil} \frac{C_\alpha(v - \lceil y\rceil + \frac{2}{N})^{\alpha}}{\alpha(\lceil y\rceil - v)^{\alpha}}\mathrm{d}v$$
$$\leq \frac{2^\alpha C_\alpha}{\alpha N^\alpha}\int_{\lceil y\rceil - \frac{1}{N}}^{\lceil y\rceil} \frac{1}{(\lceil y\rceil - v)^{\alpha}}\mathrm{d}v \leq \frac{2^\alpha C_\alpha}{\alpha N^\alpha}\frac{1}{(1-\alpha)N^{1-\alpha}} \leq \frac{1}{N} g\big(1\big)\bar{f}\big(1\big),$$

which matches with (5.166) (recall that $g(1) = g(0)$).

- Next we consider the case $x = \lceil y'\rceil$ (i.e. $x - y' = \frac{1}{N}$), and necessary $\lceil y\rceil - y \geq \frac{2}{N}$. Bounding $\lceil y\rceil - v \geq \lceil y\rceil - y - \frac{1}{N}$ and replaicing $\int_y^{y+\frac{1}{N}}$ with $\int_{\lceil y'\rceil}^{\lceil y'\rceil + \frac{2}{N}}$, since $y - x \leq \frac{1}{N}$, we get

$$\int_x^{x+\frac{1}{N}}\left(\int_y^{y+\frac{1}{N}} \frac{C_\alpha \mathbb{1}_{\{u<v\}}}{(u-v')^{1+\alpha}(v-u)^{1-\alpha}} \frac{\alpha(\lceil y'\rceil - v')^{\alpha}}{\alpha(\lceil y\rceil - v)^{\alpha}} \mathrm{d}v\right)\mathrm{d}u$$
$$\leq N^\alpha \bar{f}\Big(N(\lceil y\rceil - y)\Big)\int_{\lceil y'\rceil}^{\lceil y'\rceil + \frac{1}{N}}\left(\int_{\lceil y'\rceil}^{\lceil y'\rceil + \frac{2}{N}} \frac{C_\alpha \mathbb{1}_{\{u<v\}}\alpha(\lceil y'\rceil - v')^{\alpha}}{(u-v')^{1+\alpha}(v-u)^{1-\alpha}}\mathrm{d}v\right)\mathrm{d}u.$$

Observe that $v - u \in [(\lceil y'\rceil - u)^+, \frac{2}{N}] \subset [0, \frac{2}{N}]$, hence the integral of $(v-u)^{-(1-\alpha)}$ gives $\frac{2^\alpha}{\alpha N^\alpha}$ and we obtain

$$\int_{\lceil y'\rceil}^{\lceil y'\rceil + \frac{1}{N}}\left(\int_{\lceil y'\rceil \vee u}^{\lceil y'\rceil + \frac{2}{N}} \frac{C_\alpha \alpha(\lceil y'\rceil - v')^{\alpha}}{(u-v')^{1+\alpha}(v-u)^{1-\alpha}}\mathrm{d}v\right)\mathrm{d}u \leq \frac{2^\alpha C_\alpha}{\alpha N^\alpha}\int_{\lceil y'\rceil}^{\lceil y'\rceil + \frac{1}{N}} \frac{\alpha(\lceil y'\rceil - v')^{\alpha}}{(u-v')^{1+\alpha}}\mathrm{d}u\,.$$

Since $\int_{\lceil y' \rceil}^{\lceil y' \rceil + \frac{1}{N}} \frac{\alpha(\lceil y' \rceil - v')^\alpha}{(u-v')^{1+\alpha}} \mathrm{d}u \le \int_{v'}^{\infty} \frac{\alpha(\lceil y' \rceil - v')^\alpha}{(u-v')^{1+\alpha}} \mathrm{d}u = 1$, the upper bound is given by

$$\frac{2^{\alpha+1} C_\alpha}{\alpha N^\alpha} \le N^{-\alpha} g\Big(1\Big) \frac{f\Big(1\Big)}{\bar{f}\Big(2\Big)},$$

which matches with (5.166).

- Finally we have to consider the case in which $x - \lceil y' \rceil \ge \frac{1}{N}$ (i.e., $x - y' \ge \frac{2}{N}$) and $\lceil y \rceil - y \ge \frac{2}{N}$. Using the bounds $(u - v')^{1+\alpha} \ge (x - y' - \frac{1}{N})^{1+\alpha}$ and $\alpha(\lceil y \rceil - v)^\alpha \ge \alpha(\lceil y \rceil - y - \frac{1}{N})^\alpha$, together with $(\lceil y' \rceil - v')^\alpha \le (\lceil y' \rceil - y')^\alpha$, we can decouple the integral analogously to (5.172), obtaining

$$\int_x^{x+\frac{1}{N}} \left(\int_y^{y+\frac{1}{N}} \frac{C_\alpha \mathbb{1}_{\{u<v\}}}{(u-v')^{1+\alpha}(v-u)^{1-\alpha}} \frac{\alpha(\lceil y' \rceil - v')^\alpha}{\alpha(\lceil y \rceil - v)^\alpha} \mathrm{d}v \right) \mathrm{d}u$$
$$\le N^{1+\alpha} \frac{f\Big(N(x-y')\Big)}{\bar{f}\Big(N(\lceil y' \rceil - y') + 1\Big)} \bar{f}\Big(N(\lceil y \rceil - y)\Big) \int_{y-\frac{1}{N}}^{y+\frac{1}{N}} \left(\int_y^{y+\frac{1}{N}} \frac{C_\alpha \mathbb{1}_{\{u<v\}}}{(v-u)^{1-\alpha}} \mathrm{d}v \right) \mathrm{d}u.$$

A direct computation gives

$$\int_{y-\frac{1}{N}}^{y+\frac{1}{N}} \left(\int_y^{y+\frac{1}{N}} \frac{C_\alpha \mathbb{1}_{\{u<v\}}}{(v-u)^{1-\alpha}} \mathrm{d}v \right) \mathrm{d}u = N^{-(1+\alpha)} C_\alpha \frac{2^{1+\alpha} - 1}{\alpha(1+\alpha)} \le N^{-(1+\alpha)} g\Big(1\Big),$$

which matches with (5.166) and it concludes the proof.

5.B Miscellanea

5.B.1 Proof of Lemma 5.12

We start with the second part: assuming (5.50), we show that (5.10) holds. Given $n \in \mathbb{N}$ and a convex 1-Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$, the set $A := \{\omega \in \mathbb{R}^n : f(\omega) \le a\}$ is convex, for all $a \in \mathbb{R}$, and $\{f(\omega) \ge a + t\} \subseteq \{d(\omega, A) \ge t\}$, because f is 1-Lipschitz. Then by (5.50)

$$\mathbb{P}(f(\omega) \le a)\, \mathbb{P}(f(\omega) \ge a + t) \le \mathbb{P}(\omega \in A)\mathbb{P}(d(\omega, A) \ge t) \le C_1' \exp\left(-\frac{t^\gamma}{C_2'}\right). \tag{5.173}$$

Let $M_f \in \mathbb{R}$ be a median for $f(\omega)$, i.e. $\mathbb{P}(f(\omega) \ge M_f) \ge \frac{1}{2}$ and $\mathbb{P}(f(\omega) \le M_f) \ge \frac{1}{2}$. Applying (5.173) for $a = M_f$ and $a = M_f - t$ yields

$$\mathbb{P}\Big(\big|f(\omega) - M_f\big| \ge t\Big) \le 4\, C_1' \exp\left(-\frac{t^\gamma}{C_2'}\right),$$

which is precisely our goal (5.10).

Next we assume (5.10) and we show that (5.50) holds. We actually prove a stronger statement: for any $\eta \in (0, \infty)$

$$\mathbb{P}(\omega \in A)^\eta\, \mathbb{P}(d(\omega, A) > t) \le C_1^{1+\eta} \exp\Big(-\frac{\varepsilon_\eta t^\gamma}{C_2}\Big), \quad \text{with} \quad \varepsilon_\eta := \frac{\eta}{(1 + \eta^{\frac{1}{\gamma-1}})^{\gamma-1}}. \tag{5.174}$$

In particular, choosing $\eta = 1$, (5.50) holds with $C_1' := C_1^2$ and $C_2' = 2^{(\gamma-1)^+} C_2$. □

If A is convex, the function $f(x) := d(x, A)$ is convex, 1-Lipschitz and also $M_f \geq 0$, hence by (5.10)

$$\mathbb{P}(\omega \in A) = \mathbb{P}(f(\omega) \leq 0) \leq \mathbb{P}(|f(\omega) - M_f| \geq M_f) \leq C_1 \exp\Big(- \frac{M_f^\gamma}{C_2} \Big), \tag{5.175}$$

$$\mathbb{P}(d(\omega, A) > t) \leq \mathbb{P}(|f(\omega) - M_f| > t - M_f) \leq C_1 \exp\Big(- \frac{(t - M_f)^\gamma}{C_2} \Big), \quad \forall t \geq M_f\,, \tag{5.176}$$

hence for every $\eta \in (0, \infty)$ we obtain

$$\mathbb{P}(\omega \in A)^\eta \, \mathbb{P}(d(\omega, A) > t) \leq C_1^{1+\eta} \exp\Big(- \frac{1}{C_2} (\eta \, M_f^\gamma + (t - M_f)^\gamma) \Big), \quad \forall t \geq M_f\,. \tag{5.177}$$

The function $m \mapsto \eta \, m^\gamma + (t - m)^\gamma$ is convex and, by direct computation, it attains its minimum in the interval $[0, t]$. at the point $m = \bar{m} := t/(1 + \eta^{1/(\gamma-1)})$. Replacing M_f by $\bar{m}$ in (5.177) yields precisely (5.174) for all $t \geq M_f$.

It remains to prove (5.174) for $t \in [0, M_f)$. This follows by (5.175):

$$\mathbb{P}(\omega \in A)^\eta \, \mathbb{P}(d(\omega, A) > t) \leq \mathbb{P}(\omega \in A)^\eta \leq C_1^\eta \exp\Big(- \frac{\eta \, M_f^\gamma}{C_2} \Big) \leq C_1^{1+\eta} \exp\Big(- \frac{\varepsilon_\eta \, t^\gamma}{C_2} \Big) \quad \text{for } \ t \leq M_f\,,$$

where the last inequality holds because $\eta \geq \varepsilon_\eta$ (by (5.174)) and $C_1 \geq 1$ (by (5.10), for $t = 0$). □

5.B.2 Proof of Proposition 5.13

By convexity, $f(\omega) - f(\omega') \leq \langle \nabla f(\omega), \omega - \omega' \rangle \leq |\nabla f(\omega)| \, |\omega - \omega'|$ for all $\omega, \omega' \in \mathbb{R}^n$, where $\langle \cdot, \cdot \rangle$ is the usual scalar product in $\mathbb{R}^n$. Defining the convex set $A := \{\omega \in \mathbb{R}^n : \ f(\omega) \leq a - t\}$, we get

$$f(\omega) \leq a - t + |\nabla f(\omega)| \, |\omega - \omega'|\,, \qquad \forall \omega \in \mathbb{R}^n, \ \forall \omega' \in A\,,$$

hence $f(\omega) \leq a - t + |\nabla f(\omega)| \, d(\omega, A)$ for all $\omega \in \mathbb{R}^n$. Consequently, by inclusion of events and (5.50),

$$\mathbb{P}(f(\omega) \geq a, \ |\nabla f(\omega)| \leq c) \leq \mathbb{P}(d(\omega, A) \geq t/c) \leq \frac{C_1'}{\mathbb{P}(\omega \in A)} \exp\Big(- \frac{(t/c)^\gamma}{C_2'} \Big)\,.$$

Since $\mathbb{P}(\omega \in A) = \mathbb{P}(f(\omega) \leq a - t)$ by definition of A, we have proved (5.51). □

Continuum Free Energy, Proof Theorem 3.7

6

In this chapter we prove the existence of the continuum free energy, by proving Theorem 3.7. The proof is organized in several parts. In the first one we define a super-additive modification of $\mathbf{Z}^{W,c}_{\hat\beta,\hat h}$ and Kingman's super-additive theorem [58] gives the existence of the free energy for such modification. In the second part we deduce the existence of the free energy for $\mathbf{Z}^{W,c}_{\hat\beta,\hat h}$ and then $\mathbf{Z}^{W}_{\hat\beta,\hat h}$. The strategy that we are going to use is very similar to the one already used for the related problem regarding the continuum Copolymer model [22].

6.1 Modified partition function

In this section we aim to introduce a modified continuum partition function, $\tilde{\mathbf{Z}}^{W}_{\hat\beta,\hat h}$, for which the free energy exists, cf. Theorem 6.2 below.

Definition 6.1. We define the *modified continuum partition function* as follows:

- Given two fixed positive real numbers x, t such that $x < t-1$ we define

$$\hat{\mathbf{Z}}^{W,c}_{\hat\beta,\hat h}(x,t) = \int_x^{t-1} \mathrm{d}u \int_{t-1}^{t} \mathrm{d}v \frac{C_\alpha \mathbf{Z}^{W,c}_{\hat\beta,\hat h}(x,u)}{(u-x)^{1-\alpha}(v-u)^{1+\alpha}}, \tag{6.1}$$

  where $C_\alpha = \frac{\alpha}{\pi}\sin(\alpha\pi)$.

- For all $s \le t$, we call the *modified continuum partition function* the following process:

$$\tilde{\mathbf{Z}}^{W}_{\hat\beta,\hat h}(s,t) := \inf_{x\in[s-1,s\wedge(t-1)]} \hat{\mathbf{Z}}^{W,c}_{\hat\beta,\hat h}(x,t) \tag{6.2}$$

Theorem 6.2. *The limit*

$$\mathbf{F}^{\alpha}(\hat\beta,\hat h) := \lim_{t\to\infty} \frac{1}{t}\mathbb{E}\left[\log \tilde{\mathbf{Z}}^{W}_{\hat\beta,\hat h}(1,t)\right] = \lim_{t\to\infty}\frac{1}{t}\log \tilde{\mathbf{Z}}^{W}_{\hat\beta,\hat h}(1,t)$$

exists $\mathbb{P}$*-a.s. and in* $L^1(\mathbb{P})$

Proof. We show below that the process $\left(\log \tilde{\mathbf{Z}}^{W}_{\hat\beta,\hat h}(s,t)\right)_{1\le s\le t}$ is *stationary*, cf. (6.7), *super-additive*, cf. (6.8), and $\sup_{t\ge1}\frac{1}{t}\mathbb{E}\left(\log \tilde{\mathbf{Z}}^{W}_{\hat\beta,\hat h}(1,t)\right) < \infty$. Kingman theorem's [58] ensures that

$$\mathbf{F}^{\alpha}(\hat\beta,\hat h) := \lim_{N\to\infty}\frac{1}{N}\log \tilde{\mathbf{Z}}^{W}_{\hat\beta,\hat h}(1,N), \quad \mathbb{P}-\text{a.s. and in } L^1(\mathbb{P}). \tag{6.3}$$

Moreover we prove that such limit (6.3) exists even if we take it with respect to the continuum parameter $t \in \mathbb{R}_+$ and this concludes the proof of the theorem. Let us show how it turns out.

Let t be a fixed positive real number large enough and let n be such that $t \in [n, n+1]$. By super-additivity of $\left(\tilde{\mathbf{Z}}^{W}_{\hat\beta,\hat h}(s,t)\right)_{s\le t}$, cf. (6.8), we have

$$\log \tilde{\mathbf{Z}}^{W}_{\hat\beta,\hat h}(1,n) + \log \tilde{\mathbf{Z}}^{W}_{\hat\beta,\hat h}(n,t) \le \log \tilde{\mathbf{Z}}^{W}_{\hat\beta,\hat h}(1,t) \le \log \tilde{\mathbf{Z}}^{W}_{\hat\beta,\hat h}(1,n+1) - \log \tilde{\mathbf{Z}}^{W}_{\hat\beta,\hat h}(t,n+1). \tag{6.4}$$

By setting

$$\Omega_n = \sup_{(s,t)\in[n,n+1]^2_{\leq}} \left|\log \tilde{\mathbf{Z}}^W_{\hat{\beta},\hat{h}}(s,t)\right|, \tag{6.5}$$

we have to prove that

$$\lim_{n\to\infty} \frac{1}{n}\Omega_n = 0, \quad \mathbb{P}-\text{a.s. and in } L^1(\mathbb{P}), \tag{6.6}$$

Since $\tilde{\mathbf{Z}}^W_{\hat{\beta},\hat{h}}(s,t)$ is a stationary process, it holds that $\Omega_n \stackrel{(d)}{=} \Omega_1$. The important fact that we want to stress is that the condition $\mathbb{E}(\Omega_1) < \infty$ implies the result. On the other hand, such condition ensures that $\lim_{n\to\infty} \frac{1}{n}\mathbb{E}[\Omega_n] = 0$, that is the $L^1(\mathbb{P})$ convergence. Furthermore such condition allows to use the Borel-Cantelli's lemma to prove the a.s. convergence, indeed it holds that

$$\forall\, \varepsilon > 0 \qquad \sum_{n\in\mathbb{N}} \mathbb{P}(\Omega_n > \varepsilon n) = \sum_{n\in\mathbb{N}} \mathbb{P}(\Omega_1 > \varepsilon n) \leq \varepsilon^{-1}\int_0^\infty \mathbb{P}(\Omega_1 > t)\,\mathrm{d}t = \varepsilon^{-1}\mathbb{E}(\Omega_1) < \infty.$$

To prove that $\mathbb{E}(\Omega_1) < \infty$ we note that there exists a constant $\mathsf{c} > 0$ such that, uniformly on $(s,t) \in [1,2]^2_{\leq}$, holds that $\tilde{\mathbf{Z}}^W_{\hat{\beta},\hat{h}}(s,t) \leq \mathsf{c}\,\sup_{(x,t)\in[0,2]^2_{\leq}} \mathbf{Z}^{W,\mathsf{c}}_{\hat{\beta},\hat{h}}(x,t)$, cf.(6.1)-(6.2). Therefore, since $|\log t| \leq t + 1/t$, we get

$$\begin{aligned}\mathbb{E}(\Omega_1) = \mathbb{E}\left(\sup_{(t,s)\in[1,2]^2_{\leq}} \log\left(\tilde{\mathbf{Z}}^W_{\hat{\beta},\hat{h}}(s,t)\right)\right) &\leq \mathbb{E}\left(\log\left(\mathsf{c}\sup_{(x,t)\in[0,2]^2_{\leq}} \mathbf{Z}^{W,\mathsf{c}}_{\hat{\beta},\hat{h}}(x,t)\right)\right) \leq \\ \leq \mathsf{c}\,\mathbb{E}\left(\sup_{(x,t)\in[0,2]^2_{\leq}} \mathbf{Z}^{W,\mathsf{c}}_{\hat{\beta},\hat{h}}(x,t)\right) &+ \mathsf{c}^{-1}\,\mathbb{E}\left(\sup_{(s,t)\in[0,2]^2_{\leq}} \frac{1}{\mathbf{Z}^{W,\mathsf{c}}_{\hat{\beta},\hat{h}}(x,t)}\right) \stackrel{\text{Prop.3.12}}{<} \infty.\end{aligned}$$

To conclude the proof we have to show that $\left(\log \tilde{\mathbf{Z}}^W_{\hat{\beta},\hat{h}}(s,t)\right)_{1\leq s\leq t}$ is *stationary*, *super-additive* and $\sup_{t\geq 1} \frac{1}{t}\mathbb{E}\left(\log \tilde{\mathbf{Z}}^W_{\hat{\beta},\hat{h}}(1,t)\right) < \infty$.

Stationary means that for any $a > 0$ and $s_i < t_i$, $i = 1,\cdots,k$ one has

$$\left(\tilde{\mathbf{Z}}^W_{\hat{\beta},\hat{h}}(s_1+a,t_1+a),\cdots,\tilde{\mathbf{Z}}^W_{\hat{\beta},\hat{h}}(s_k+a,t_k+a)\right) \stackrel{(d)}{=} \left(\tilde{\mathbf{Z}}^W_{\hat{\beta},\hat{h}}(s_1,t_1),\cdots,\tilde{\mathbf{Z}}^W_{\hat{\beta},\hat{h}}(s_k,t_k)\right). \tag{6.7}$$

Such property follows by observing that such property holds for $\mathbf{Z}^{W,\mathsf{c}}_{\hat{\beta},\hat{h}}$ (the Weiner Chaos expansion which defines $\mathbf{Z}^{W,\mathsf{c}}_{\hat{\beta},\hat{h}}$, (2.46), is stationary) and thus it must hold for $\tilde{\mathbf{Z}}^W_{\hat{\beta},\hat{h}}$.

We show that the process $\left(\log \tilde{\mathbf{Z}}^W_{\hat{\beta},\hat{h}}(s,t)\right)_{1\leq s\leq t}$ is super-additive, cf.(6.1)-(6.2). Precisely we prove that for any $1 \leq r < s < t < \infty$

$$\tilde{\mathbf{Z}}^W_{\hat{\beta},\hat{h}}(r,t) \geq \tilde{\mathbf{Z}}^W_{\hat{\beta},\hat{h}}(r,s)\cdot\tilde{\mathbf{Z}}^W_{\hat{\beta},\hat{h}}(s,t). \tag{6.8}$$

For any fixed $x \in [r-1, r\wedge(s-1)]$, where $s \in (r,t)$, we can decompose $\hat{\mathbf{Z}}^{W,\mathsf{c}}_{\hat{\beta},\hat{h}}(x,t)$ in the sum of two integrals by splitting the integral $\int_x^{t-1}$ with respect to the point $s-1$. Then we get a lower bound replacing first internal integral $\int_{t-1}^{t}$ by $\int_{t-1}^{s\wedge(t-1)}$, obtaining a lower bound, precisely

$$\hat{\mathbf{Z}}^{W,\mathsf{c}}_{\hat{\beta},\hat{h}}(x,t) \geq \int_x^{s-1}\mathrm{d}u\int_{t-1}^{s\wedge(t-1)}\mathrm{d}v\frac{C_\alpha\mathbf{Z}^{W,\mathsf{c}}_{\hat{\beta},\hat{h}}(x,u)}{(u-x)^{1-\alpha}(v-u)^{1+\alpha}} + \int_{s-1}^{t-1}\mathrm{d}u\int_{t-1}^{t}\mathrm{d}v\frac{C_\alpha\mathbf{Z}^{W,\mathsf{c}}_{\hat{\beta},\hat{h}}(x,u)}{(u-x)^{1-\alpha}(v-u)^{1+\alpha}}, \tag{6.9}$$

with the convention that whenever $[s-1,s]$ and $[t-1,t]$ have empty intersection the first integral is assumed to be zero.

Let us consider the second integral. We decompose $\mathbf{Z}^{W,\mathtt{c}}_{\hat\beta,\hat h}(x,u)$ by using [24, (iv), Theorem 2.4] with respect to the point $s-1$:

$$\begin{aligned}&\int_{s-1}^{t-1}\mathrm{d}u\int_{t-1}^{t}\mathrm{d}v\frac{C_\alpha \mathbf{Z}^{W,\mathtt{c}}_{\hat\beta,\hat h}(x,u)}{(u-x)^{1-\alpha}(v-u)^{1+\alpha}}\\ &=\int_{s-1}^{t-1}\mathrm{d}u\int_{t-1}^{t}\mathrm{d}v\frac{1}{(v-u)^{1+\alpha}}\int_{x}^{s-1}\mathrm{d}y\int_{s-1}^{s}\mathrm{d}z\frac{C_\alpha \mathbf{Z}^{W,\mathtt{c}}_{\hat\beta,\hat h}(x,y)C_\alpha \mathbf{Z}^{W,\mathtt{c}}_{\hat\beta,\hat h}(z,u)}{(y-x)^{1-\alpha}(z-y)^{1+\alpha}(u-z)^{1-\alpha}}\mathbb{1}_{\{z<u\}}.\end{aligned}\tag{6.10}$$

We interchange the integrals. Since $z<u$ we obtain

$$\begin{aligned}&\left(\int_{x}^{s-1}\mathrm{d}y\int_{s-1}^{s\wedge(t-1)}\mathrm{d}z\frac{C_\alpha \mathbf{Z}^{W,\mathtt{c}}_{\hat\beta,\hat h}(x,y)}{(y-x)^{1-\alpha}(z-y)^{1+\alpha}}\left(\int_{z}^{t-1}\mathrm{d}u\int_{t-1}^{t}\mathrm{d}v\frac{C_\alpha \mathbf{Z}^{W,\mathtt{c}}_{\hat\beta,\hat h}(z,u)}{(u-z)^{1-\alpha}(v-u)^{1+\alpha}}\right)\right)\\ &\geq\left(\int_{x}^{s-1}\mathrm{d}y\int_{s-1}^{s\wedge(t-1)}\mathrm{d}z\frac{C_\alpha \mathbf{Z}^{W,\mathtt{c}}_{\hat\beta,\hat h}(x,y)}{(y-x)^{1-\alpha}(z-y)^{1+\alpha}}\right)\tilde{\mathbf{Z}}^{W}_{\hat\beta,\hat h}(s,t).\end{aligned}\tag{6.11}$$

In particular if $s<t-1$ we obtain immediately that, cf. (6.9),

$$\hat{\mathbf{Z}}^{W,\mathtt{c}}_{\hat\beta,\hat h}(x,t)\geq\hat{\mathbf{Z}}^{W,\mathtt{c}}_{\hat\beta,\hat h}(x,s)\tilde{\mathbf{Z}}^{W}_{\hat\beta,\hat h}(s,t)$$

because the first integral is equal to 0.

If $s\geq t-1$ we have to take in account also the first integral. In this case $1\geq\tilde{\mathbf{Z}}^{W}_{\hat\beta,\hat h}(s,t)$ because, by explicit computations, cf. (6.1),

$$\lim_{x\to t-1}\hat{\mathbf{Z}}^{W,\mathtt{c}}_{\hat\beta,\hat h}(x,t)=1\tag{6.12}$$

and therefore, together with (6.11),

$$\begin{aligned}&\hat{\mathbf{Z}}^{W,\mathtt{c}}_{\hat\beta,\hat h}(x,t)\geq\\ &\quad\int_{x}^{s-1}\mathrm{d}u\int_{t-1}^{s}\mathrm{d}v\frac{C_\alpha \mathbf{Z}^{W,\mathtt{c}}_{\hat\beta,\hat h}(x,u)}{(u-x)^{1-\alpha}(v-u)^{1+\alpha}}+\tilde{\mathbf{Z}}^{W}_{\hat\beta,\hat h}(s,t)\int_{x}^{s-1}\mathrm{d}u\int_{s-1}^{t-1}\mathrm{d}v\frac{C_\alpha \mathbf{Z}^{W,\mathtt{c}}_{\hat\beta,\hat h}(x,u)}{(u-x)^{1-\alpha}(v-u)^{1+\alpha}}\\ &\quad\geq\tilde{\mathbf{Z}}^{W}_{\hat\beta,\hat h}(s,t)\int_{x}^{s-1}\mathrm{d}u\int_{s-1}^{s}\mathrm{d}v\frac{C_\alpha \mathbf{Z}^{W,\mathtt{c}}_{\hat\beta,\hat h}(x,u)}{(u-x)^{1-\alpha}(v-u)^{1+\alpha}}=\hat{\mathbf{Z}}^{W,\mathtt{c}}_{\hat\beta,\hat h}(x,s)\tilde{\mathbf{Z}}^{W}_{\hat\beta,\hat h}(s,t).\end{aligned}$$

By taking the inf on all possibles values of x we obtain the result.

Finally we prove that

$$\sup_{t\geq 1}\frac{1}{t}\mathbb{E}\left[\log\tilde{\mathbf{Z}}^{W}_{\hat\beta,\hat h}(1,t)\right]<\infty.\tag{6.13}$$

According to (6.12), $\tilde{\mathbf{Z}}^{W}_{\hat\beta,\hat h}(1,t)\leq 1$ whenever $t\in(1,2)$, therefore it is enough to consider $t\in(2,+\infty)$. In this case, Jensen's inequality implies $\mathbb{E}\left[\log\tilde{\mathbf{Z}}^{W}_{\hat\beta,\hat h}(1,t)\right]\leq\log\mathbb{E}\left[\tilde{\mathbf{Z}}^{W}_{\hat\beta,\hat h}(1,t)\right]$. To conclude the proof we are going to show that $\mathbb{E}\left[\tilde{\mathbf{Z}}^{W}_{\hat\beta,\hat h}(1,t)\right]\leq C\,e^{c\,t}$, for some $c,C>0$. By definition

$$\mathbb{E}\left[\tilde{\mathbf{Z}}^{W}_{\hat\beta,\hat h}(1,t)\right]\leq\mathbb{E}\left[\hat{\mathbf{Z}}^{W,\mathtt{c}}_{\hat\beta,\hat h}(1,t)\right]=\int_{1}^{t-1}\mathrm{d}u\int_{t-1}^{t}\mathrm{d}v\frac{C_\alpha\mathbb{E}\left[\mathbf{Z}^{W,\mathtt{c}}_{\hat\beta,\hat h}(1,u)\right]}{(u-1)^{1-\alpha}(v-u)^{1+\alpha}}.\tag{6.14}$$

We want to show that $\mathbb{E}\left[\mathbf{Z}^{W,\mathtt{c}}_{\hat\beta,\hat h}(1,u)\right]$ is dominated by an exponential factor. By [23, Section 2.3.1] and

Cauchy-Schwarz inequality can relate the partition function, $\mathbf{Z}^{W,\mathsf{c}}_{\hat\beta,\hat h}$, with $\hat h \neq 0$ to the one with $\hat h = 0$ through an explicit Radon-Nikodym derivative:

$$\mathbb{E}\left[\mathbf{Z}^{W,\mathsf{c}}_{\hat\beta,\hat h}(1,u)\right] = \mathbb{E}\left[\mathbf{Z}^{W,\mathsf{c}}_{\hat\beta,0}(1,u)e^{\frac{\hat h}{\hat\beta}W_u - \frac{\hat h^2}{2\hat\beta^2}u}\right] \le \mathbb{E}\left[\mathbf{Z}^{W,\mathsf{c}}_{\hat\beta,0}(1,u)^2\right]^{\frac{1}{2}} e^{\frac{\hat h^2}{2\hat\beta^2}u}. \tag{6.15}$$

We have to estimate $\mathbb{E}\left[\mathbf{Z}^{W,\mathsf{c}}_{\hat\beta,0}(1,u)^2\right]$. By [24, Appendix C] and the theory of the Wiener Chaos expansion, see (2.46) and [54], it holds that

$$\mathbb{E}\left[\mathbf{Z}^{W,\mathsf{c}}_{\hat\beta,0}(1,u)^2\right] = \sum_{k=0}^{\infty} \int\cdots\int_{1<u_1<\cdots<u_k<u} \frac{C_\alpha^{2k}\,\hat\beta^{2k}\,(u-1)^{2(1-\alpha)}}{(u_1-1)^{2(1-\alpha)}\cdots(u-u_k)^{2(1-\alpha)}}dt_1\,\cdots\,dt_k$$

$$= \sum_{k=0}^{\infty} C_\alpha^{2k}\,\hat\beta^{2k}\,(u-1)^{(2\alpha-1)k}\frac{\Gamma(2\alpha-1)^{k+1}}{\Gamma((k+1)(2\alpha-1))}.$$

Note that $2\alpha - 1 \in (0,1)$, then our problem has reduced to prove that for any constant $c \in (0,1)$ there exist $c_1, c_2 > 0$ such that

$$f(t) = \sum_{k=0}^{\infty}\frac{t^{c\,k}}{\Gamma(c\,k)} \le c_1 e^{c_2 t}. \tag{6.16}$$

By Stirling's formula there exist $\theta \in (0,1)$ such that $\Gamma(x) \overset{x\to\infty}{\sim} \sqrt{2\pi}x^{x-\frac{1}{2}}e^{-x+\frac{\theta}{12x}}$, from which we deduce that there exists a constant $c_0 > 0$ for which $\Gamma(ck) \ge c_0^{-k}\,k^{ck}$, for all $k \in \mathbb{N}$. This implies that

$$f(t) \le \sum_{k=0}^{\infty}\left(\frac{(c_0 t)^k}{k^k}\right)^c.$$

Now we observe that for any fixed $A > 0$ there exists $c_A > 0$ such that $k^k \ge c_A\,k!\,k^A$ for all $k > 0$. Hölder inequality with $1/p = c$ gives

$$\sum_{k=0}^{\infty}\left(\frac{(c_0 t)^k}{k^k}\right)^c \le \left(\sum_{k=1}^{\infty}\frac{1}{(c_A\,k)^{\frac{A}{1-c}}}\right)^{1-c}\left(\sum_{k=0}^{\infty}\frac{(c_0 t)^k}{k!}\right)^c = Ce^{c_0\,c\,t},$$

for a suitable choice of $A > 0$ and some constant $C > 0$. □

6.2 Existence of the Free Energy, conditioned case

In this section we aim at showing how to deduce the existence of the free energy for the conditional continuum partition function $\mathbf{Z}^{W,\mathsf{c}}_{\hat\beta,\hat h}(0,t)$ from Theorem 6.2, which ensures the existence of the free energy for the modified partition function $\tilde{\mathbf{Z}}^{W}_{\hat\beta,\hat h}(1,t)$. In particular we prove the following theorem

Theorem 6.3. *For any $t \ge 1$*

$$c_t^{(1)}\tilde{\mathbf{Z}}^{W}_{\hat\beta,\hat h}(1,t) \le \mathbf{Z}^{W,\mathsf{c}}_{\hat\beta,\hat h}(0,t) \le c_t^{(2)}\tilde{\mathbf{Z}}^{W}_{\hat\beta,\hat h}(1,t), \tag{6.17}$$

where $c_t^{(1)}$ and $c_t^{(2)}$ are two stochastic processes such that $\lim_{t\to\infty}\frac{1}{t}\log c_t^{(i)} = 0$, $\mathbb{P}$-a.s. and in $L^1(\mathbb{P})$, for $i = 1,2$.

Proof. In the first part of the proof we are going to prove the lower bound. In particular we prove that

$$c_t := \inf_{v\in[t-1,t]}\mathbf{Z}^{W,\mathsf{c}}_{\hat\beta,\hat h}(v,t), \quad t \in (1,\infty). \tag{6.18}$$

We want to show that we can choose $c_1^{(1)} = c_t$. For this purpose we start to show that it gives the lower bound. To make this we decompose $\mathbf{Z}_{\hat{\beta},\hat{h}}^{W,\mathrm{c}}(0,t)$ according to [24, (iv), theorem 2.4],

$$\mathbf{Z}_{\hat{\beta},\hat{h}}^{W,\mathrm{c}}(0,t) = \frac{t^{1-\alpha}}{C_\alpha} \int_0^{t-1} \mathrm{d}u \int_{t-1}^{t} \mathrm{d}v \frac{C_\alpha \mathbf{Z}_{\hat{\beta},\hat{h}}^{W,\mathrm{c}}(0,u) \cdot C_\alpha \mathbf{Z}_{\hat{\beta},\hat{h}}^{W,\mathrm{c}}(v,t)}{u^{1-\alpha}(v-u)^{1+\alpha}(t-v)^{1-\alpha}}, \tag{6.19}$$

and by observing that $t - v \leq 1$ and $t \geq 1$, we conclude $\mathbf{Z}_{\hat{\beta},\hat{h}}^{W,\mathrm{c}}(0,t) \geq c_t \tilde{\mathbf{Z}}_{\hat{\beta},\hat{h}}^{W}(1,t)$, cf. (6.2).

We have to prove that $\frac{1}{t}\log c_t \to 0$ as $t \to \infty$, $\mathbb{P}$-a.s. and in $L^1(\mathbb{P})$. Note that c_t is a stationary process because $\mathbf{Z}_{\hat{\beta},\hat{h}}^{W,\mathrm{c}}(v,t)$ is, therefore to have the $L^1(\mathbb{P})$ - convergence we have to prove that $\mathbb{E}\left[\left|\log c_1\right|\right] < \infty$. For this purpose we observe that $|\log t| \leq t + 1/t$ and thus the fact that $\mathbb{E}\left[\left|\log c_1\right|\right] < \infty$ follows by Proposition 3.12.

To prove the a.s.-convergence, for any $\varepsilon > 0$ and $n \in \mathbb{N}$ we consider the event

$$\left\{\exists\, t \in [n, n+1] \,:\, \left|\frac{1}{n}\log c_t\right| = \frac{1}{n}\log\frac{1}{c_t} \geq \varepsilon\right\} = \left\{\sup_{t\in[n,n+1]} \frac{1}{c_t} \geq e^{\varepsilon n}\right\}, \tag{6.20}$$

(recall that $c_t \leq 1$ for any $t \geq 1$, because $\mathbf{Z}_{\hat{\beta},\hat{h}}^{W,\mathrm{c}}(t,t) = 1$) and we denote by p_n its probability. Then, if $\sum_n p_n < \infty$, Borel-Cantelli lemma ensures that $\lim_{t\to\infty} \frac{1}{t}\log c_t = 0$ with probability 1. To prove this claim we observe that Markov's inequality gives

$$p_n \leq e^{-\varepsilon n}\mathbb{E}\left(\sup_{t\in[n,n+1]} \frac{1}{c_t}\right),$$

and thus we conclude the proof by showing that $\mathbb{E}\left(\sup_{t\in[n,n+1]} \frac{1}{c_t}\right)$ is bounded by a constant independent of n. This follows by Proposition 3.12 by observing that

$$\sup_{t\in[n,n+1]} \frac{1}{c_t} = \sup_{t\in[n,n+1]}\left(\sup_{v\in[t-1,t]} \frac{1}{\mathbf{Z}_{\hat{\beta},\hat{h}}^{W,\mathrm{c}}(v,t)}\right) \leq \sup_{(t,v)\in[n,n+1]\times[n-1,n]} \frac{1}{\mathbf{Z}_{\hat{\beta},\hat{h}}^{W,\mathrm{c}}(v,t)} =: \phi_t$$

and that ϕ_t is stationary.

For the upper bound we use the same strategy used to get the lower bound in (6.19). In analogy with c_t we consider

$$k_t = \sup_{v\in[t-1,t]} \mathbf{Z}_{\hat{\beta},\hat{h}}^{W,\mathrm{c}}(v,t) \tag{6.21}$$

and, as well as c_t, it holds that $\frac{1}{t}\log k_t \to 0$ as $t \to \infty$, $\mathbb{P}$-a.s. and in $L^1(\mathbb{P})$. Also in this case the idea is to get an upper bound on (6.19) by recovering $\tilde{\mathbf{Z}}_{\hat{\beta},\hat{h}}^{W}(1,t)$. Unfortunately in this case the factor $t - v$ cannot be bounded so easy; we need some intermediate result. Let us start with a technical lemma:

Lemma 6.4. *For any fixed $\alpha \in (1/2, 1)$ there exists a constant ξ_α such that, uniformly on $u < t - 1$, $t > 2$*

$$\frac{\displaystyle\int_{t-1}^{t} \frac{\mathrm{d}v}{(v-u)^{1+\alpha}(t-v)^{1-\alpha}}}{\displaystyle\int_{t-1}^{t} \frac{\mathrm{d}v}{(v-u)^{1+\alpha}}} \leq \xi_\alpha. \tag{6.22}$$

Proof. The proof follows by a direct computations: first we split the integral in two,

$$\int_{t-1}^{t} \frac{\mathrm{d}v}{(v-u)^{1+\alpha}(t-v)^{1-\alpha}} = \int_{t-1}^{t-1/2} \frac{\mathrm{d}v}{(v-u)^{1+\alpha}(t-v)^{1-\alpha}} + \int_{t-1/2}^{t} \frac{\mathrm{d}v}{(v-u)^{1+\alpha}(t-v)^{1-\alpha}}.$$

In the first integral we bound $t-v \geq \frac{1}{2}$ and then we replace $\int_{t-1}^{t-\frac{1}{2}}$ by $\int_{t-1}^{t}$, getting an upper bound, namely

$$\int_{t-1}^{t-1/2} \frac{\mathrm{d}v}{(v-u)^{1+\alpha}(t-v)^{1-\alpha}} \leq 2^{1-\alpha} \int_{t-1}^{t} \frac{\mathrm{d}v}{(v-u)^{1+\alpha}}.$$

In the second one we bound $v-u \geq t-\frac{1}{2}-u$, obtaining

$$\int_{t-1/2}^{t} \frac{\mathrm{d}v}{(v-u)^{1+\alpha}(t-v)^{1-\alpha}} \leq \frac{2^{\alpha}}{\alpha} \frac{1}{(t-\frac{1}{2}-u)^{1+\alpha}}.$$

We conclude by observing that for $u < t-1$ we can replace $\frac{1}{(t-\frac{1}{2}-u)^{1+\alpha}}$ with $\frac{1}{(t-u)^{1+\alpha}}$ by paying a positive constant ξ'_α depending only on α. This conclude the proof because $\frac{1}{(t-u)^{1+\alpha}} \leq \int_{t-1}^{t} \mathrm{d}v \frac{1}{(v-u)^{1+\alpha}}$. □

For any fixed $x \in [0,1]$ we can decompose $\mathbf{Z}_{\hat\beta,\hat h}^{W,\mathrm{c}}(x,t)$ as in (6.19)

$$\mathbf{Z}_{\hat\beta,\hat h}^{W,\mathrm{c}}(x,t) = \frac{(t-x)^{1-\alpha}}{C_\alpha} \int_x^{t-1} \mathrm{d}u \int_{t-1}^{t} \mathrm{d}v \frac{C_\alpha \mathbf{Z}_{\hat\beta,\hat h}^{W,\mathrm{c}}(x,u) \cdot C_\alpha \mathbf{Z}_{\hat\beta,\hat h}^{W,\mathrm{c}}(v,t)}{(u-x)^{1-\alpha}(v-u)^{1+\alpha}(t-v)^{1-\alpha}}. \tag{6.23}$$

By using the definition of k_t, (6.21) and Lemma 6.4 we get the following upper bound

$$\mathbf{Z}_{\hat\beta,\hat h}^{W,\mathrm{c}}(x,t) \leq \xi_\alpha t^{1-\alpha}\, k_t\, \hat{\mathbf{Z}}_{\hat\beta,\hat h}^{W,\mathrm{c}}(x,t).$$

In particular

$$\inf_{x\in[0,1]} \mathbf{Z}_{\hat\beta,\hat h}^{W,\mathrm{c}}(x,t) \leq \xi_\alpha t^{1-\alpha}\, k_t\, \tilde{\mathbf{Z}}_{\hat\beta,\hat h}^{W}(1,t). \tag{6.24}$$

To have an upper bound for $\mathbf{Z}_{\hat\beta,\hat h}^{W,\mathrm{c}}(0,t)$ we are going to show that there exists a (random) constant $C > 0$, independent of t, such that $\log C$ is integrable and

$$\inf_{x\in[0,1]} \mathbf{Z}_{\hat\beta,\hat h}^{W,\mathrm{c}}(x,t) \geq C\, \mathbf{Z}_{\hat\beta,\hat h}^{W,\mathrm{c}}(0,t). \tag{6.25}$$

In this case we conclude the proof by defining

$$c_t^{(2)} := \frac{\xi_\alpha}{C} t^{1-\alpha} k_t. \tag{6.26}$$

The first step to prove (6.25) we observe that a.s.

$$\begin{aligned} l &= \inf_{u,v\in[0,2]^2_\leq} \mathbf{Z}_{\hat\beta,\hat h}^{W,\mathrm{c}}(s,t) > 0, \\ L &= \sup_{u,v\in[0,2]^2_\leq} \mathbf{Z}_{\hat\beta,\hat h}^{W,\mathrm{c}}(s,t) < \infty, \end{aligned} \tag{6.27}$$

are well defined because $\mathbf{Z}_{\hat\beta,\hat h}^{W,\mathrm{c}}(u,v)$ is a continuous and strictly positive process, cf. [24, theorem 2.4]. Moreover Proposition 3.12 ensures $\mathbb{E}(|\log L|)$, $\mathbb{E}(|\log l|) < \infty$.

The second step to get (6.25) goes as follows: let $x \in [0,1]$ and $t \geq 2$ be fixed. By using the decomposition of $\mathbf{Z}^{W,c}_{\hat\beta,\hat h}(0,t)$ and $\mathbf{Z}^{W,c}_{\hat\beta,\hat h}(x,t)$ like as in (6.19), we obtain

$$\frac{\mathbf{Z}^{W,c}_{\hat\beta,\hat h}(x,t)}{\mathbf{Z}^{W,c}_{\hat\beta,\hat h}(0,t)} \ge \frac{l(t-1)^{1-\alpha}}{Lt^{1-\alpha}} \frac{\displaystyle\int_2^t \mathrm{d}v \left[\int_x^2 \frac{\mathrm{d}u}{(u-x)^{1-\alpha}(v-u)^{1+\alpha}}\right] \frac{C_\alpha \mathbf{Z}^{W,c}_{\hat\beta,\hat h}(v,t)}{(t-v)^{1-\alpha}}}{\displaystyle\int_2^t \mathrm{d}v \left[\int_0^2 \frac{\mathrm{d}u}{u^{1-\alpha}(v-u)^{1+\alpha}}\right] \frac{C_\alpha \mathbf{Z}^{W,c}_{\hat\beta,\hat h}(v,t)}{(t-v)^{1-\alpha}}}. \tag{6.28}$$

The proof follows because there exists a constant λ_α such that uniformly on $x \in [0,1]$ and $v \ge 2$

$$\frac{\displaystyle\int_x^2 \frac{\mathrm{d}u}{(u-x)^{1-\alpha}(v-u)^{1+\alpha}}}{\displaystyle\int_0^2 \frac{\mathrm{d}u}{u^{1-\alpha}(v-u)^{1+\alpha}}} \ge \lambda_\alpha. \tag{6.29}$$

In particular we conclude that

$$\frac{\inf_{x\in[0,1]} \mathbf{Z}^{W,c}_{\hat\beta,\hat h}(x,t)}{\mathbf{Z}^{W,c}_{\hat\beta,\hat h}(0,t)} \ge \frac{\lambda_\alpha}{2^{1-\alpha}} \frac{l}{L} =: C. \tag{6.30}$$

To prove (6.29) we note that for any $x \in [0,1]$

$$\frac{\displaystyle\int_x^2 \frac{\mathrm{d}u}{(u-x)^{1-\alpha}(v-u)^{1+\alpha}}}{\displaystyle\int_0^2 \frac{\mathrm{d}u}{u^{1-\alpha}(v-u)^{1+\alpha}}} \ge \frac{\displaystyle\int_1^2 \frac{\mathrm{d}u}{u^{1-\alpha}(v-u)^{1+\alpha}}}{\displaystyle\int_0^2 \frac{\mathrm{d}u}{u^{1-\alpha}(v-u)^{1+\alpha}}} =: \phi(v)$$

and the function $\phi(v)$ is continuous and strictly positive on $(2,\infty)$; $\phi(v) \underset{v\to 2^+}{\sim} 1$, and $\phi(v) \underset{v\to\infty}{\sim} \frac{2^\alpha-1}{2^\alpha}$. This implies that $\lambda_\alpha := \min_{v\in(2,\infty)} \phi(v) > 0$. □

6.3 Existence of the Free Energy, free case

In this last section of the chapter we deduce Theorem 3.7: the free energy of the continuum pinning model is well defined and it coincides with the free energy of the conditional continuum pinning model, i.e.

$$\mathbf{F}^\alpha(\hat\beta,\hat h) = \lim_{t\to\infty} \frac{1}{t} \log \mathbf{Z}^W_{\hat\beta,\hat h}(t), \quad \mathbb{P}(dW)\text{-a.s. and in } L^1(\mathbb{P}), \tag{6.31}$$

The proof follows from a sandwich argument, by getting an upper and lower bound on $\mathbf{Z}^W_{\hat\beta,\hat h}$ by using $\mathbf{Z}^{W,c}_{\hat\beta,\hat h}$ and $\tilde{\mathbf{Z}}^W_{\hat\beta,\hat h}$.

Proof. We start to observe (see Remark 6.7 below) that the continuum free partition function can be obtained by the conditioned one, by integrating over the last visited point before t by the regenerative set, $\mathsf{g}_t = \mathsf{g}_t(\boldsymbol{\tau}^\alpha)$, cf. (2.79)-(2.90):

$$\mathbf{Z}^W_{\hat\beta,\hat h}(t) = \mathrm{E}\left[\mathbf{Z}^{W,c}_{\hat\beta,\hat h}(0,\mathsf{g}_t)\right] = \int_0^t \mathrm{d}u \frac{C_\alpha}{\alpha} \frac{\mathbf{Z}^{W,c}_{\hat\beta,\hat h}(0,u)}{u^{1-\alpha}(t-u)^\alpha} = \int_0^t \mathrm{d}u \int_t^\infty \mathrm{d}v\, C_\alpha \frac{\mathbf{Z}^{W,c}_{\hat\beta,\hat h}(0,u)}{u^{1-\alpha}(v-u)^{1+\alpha}}. \tag{6.32}$$

To get the lower bound, we observe that in (6.32) we can replace $\int\limits_t^\infty$ by $\int\limits_t^{t-1}$, and by definition of $\hat{\mathbf{Z}}^{W,\mathrm{c}}_{\hat\beta,\hat h}$ and $\tilde{\mathbf{Z}}^{W}_{\hat\beta,\hat h}$, cf. (6.1) - (6.2), we have immediately that

$$\mathbf{Z}^{W}_{\hat\beta,\hat h}(t) \geq \tilde{\mathbf{Z}}^{W}_{\hat\beta,\hat h}(1, t+1), \tag{6.33}$$

which gives the lower bound.

On the other hand, to get the upper bound we need to be more careful. Let $\varepsilon \in (0, \frac{1}{4})$ be fixed. We decompose

$$\mathrm{E}\Big[\mathbf{Z}^{W,\mathrm{c}}_{\hat\beta,\hat h}(0, \mathsf{g}_t)\Big] = \mathrm{E}\Big[\mathbf{Z}^{W,\mathrm{c}}_{\hat\beta,\hat h}(0, \mathsf{g}_t)\mathbb{1}_{\mathsf{g}_t\in(0,t-\varepsilon]}\Big] + \mathrm{E}\Big[\mathbf{Z}^{W,\mathrm{c}}_{\hat\beta,\hat h}(0, \mathsf{g}_t)\mathbb{1}_{\mathsf{g}_t\in(t-\varepsilon,t)}\Big]. \tag{6.34}$$

The first term can be bounded by considering the process c_t introduced in (6.18). Indeed by decomposing $\mathbf{Z}^{W,\mathrm{c}}_{\hat\beta,\hat h}(0, t)$ according to [24, (iv), theorem 2.4], we have

$$\begin{aligned}\mathbf{Z}^{W,\mathrm{c}}_{\hat\beta,\hat h}(0, t) &= t^{1-\alpha}\int\limits_0^{t-\varepsilon}\mathrm{d}u\int\limits_{t-\varepsilon}^{t}\mathrm{d}v C_\alpha \frac{\mathbf{Z}^{W,\mathrm{c}}_{\hat\beta,\hat h}(0, u)\mathbf{Z}^{W,\mathrm{c}}_{\hat\beta,\hat h}(v, t)}{u^{1-\alpha}(v-u)^{1+\alpha}(t-v)^{1-\alpha}}\\ &\geq t^{1-\alpha}\, c_t \int_0^{t-\varepsilon} C_\alpha \frac{\mathbf{Z}^{W,\mathrm{c}}_{\hat\beta,\hat h}(0, u)}{u^{1-\alpha}}\mathrm{d}u \int\limits_{t-\varepsilon}^{t}\frac{\mathrm{d}v}{(v-u)^{1+\alpha}(t-v)^{1-\alpha}}\end{aligned}$$

To get a lower bound for $\int\limits_{t-\varepsilon}^{t}\frac{\mathrm{d}v}{(v-u)^{1+\alpha}(t-v)^{1-\alpha}}$ we observe that $(v-u)^{1+\alpha} \leq (t-u)^{1+\alpha} \leq t\,(t-u)^{\alpha}$ and the rest of the integral gives $\alpha^{-1}\varepsilon^{\alpha}$. Therefore, cf. (2.90),

$$\mathbf{Z}^{W,\mathrm{c}}_{\hat\beta,\hat h}(0, t) \geq \frac{c_t\varepsilon^\alpha}{\alpha\, t^\alpha}\int_0^{t-\varepsilon} C_\alpha \frac{\mathbf{Z}^{W,\mathrm{c}}_{\hat\beta,\hat h}(0, u)}{u^{1-\alpha}(t-u)^\alpha} = \frac{c_t\varepsilon^\alpha}{\alpha\, t^\alpha}\mathrm{E}\Big[\mathbf{Z}^{W,\mathrm{c}}_{\hat\beta,\hat h}(0, \mathsf{g}_t)\mathbb{1}_{\{\mathsf{g}_t\leq t-\varepsilon\}}\Big]. \tag{6.35}$$

In particular

$$\mathrm{E}\Big[\mathbf{Z}^{W,\mathrm{c}}_{\hat\beta,\hat h}(0, \mathsf{g}_t)\mathbb{1}_{\{\mathsf{g}_t\leq t-\varepsilon\}}\Big] \;\leq\; \frac{\alpha\, t^\alpha}{c_t\, \varepsilon^\alpha}\sup_{r\in[t-\varepsilon,t]}\mathbf{Z}^{W,\mathrm{c}}_{\hat\beta,\hat h}(0, r). \tag{6.36}$$

For the second term in (6.34), we observe that on the event $\{\mathsf{g}_t \in (t-\varepsilon, t)\}$ it holds that $\mathbf{Z}^{W,\mathrm{c}}_{\hat\beta,\hat h}(0, \mathsf{g}_t) \leq \sup_{r\in[t-\varepsilon,t]}\mathbf{Z}^{W,\mathrm{c}}_{\hat\beta,\hat h}(0, r)$ and thus

$$\mathrm{E}\Big[\mathbf{Z}^{W,\mathrm{c}}_{\hat\beta,\hat h}(0, \mathsf{g}_t)\mathbb{1}_{\mathsf{g}_t\in(t-\varepsilon,t)}\Big] \leq \sup_{r\in[t-\varepsilon,t]}\mathbf{Z}^{W,\mathrm{c}}_{\hat\beta,\hat h}(0, r)\mathrm{P}(\mathsf{g}_t \in (t-\varepsilon, t)) \leq \sup_{r\in[t-\varepsilon,t]}\mathbf{Z}^{W,\mathrm{c}}_{\hat\beta,\hat h}(0, r). \tag{6.37}$$

We highlight that also in this case we have an estimation like (6.36):

$$\mathrm{E}\Big[\mathbf{Z}^{W,\mathrm{c}}_{\hat\beta,\hat h}(0, \mathsf{g}_t)\mathbb{1}_{\mathsf{g}_t\in(t-\varepsilon,t)}\Big] \;\leq\; \sup_{r\in[t-\varepsilon,t]}\mathbf{Z}^{W,\mathrm{c}}_{\hat\beta,\hat h}(0, r). \tag{6.38}$$

Putting the estimations (6.36) and (6.38) in (6.34) we obtain

$$\mathbf{Z}^{W}_{\hat\beta,\hat h}(t) \overset{(6.32)}{=} \mathrm{E}\Big[\mathbf{Z}^{W,\mathrm{c}}_{\hat\beta,\hat h}(0, \mathsf{g}_t)\Big] \;\leq\; \left(\frac{\alpha\, t^\alpha}{c_t\, \varepsilon^\alpha} \;+\; 1\right)\cdot \sup_{r\in[t-\varepsilon,t]}\mathbf{Z}^{W,\mathrm{c}}_{\hat\beta,\hat h}(0, r). \tag{6.39}$$

We are going to show that $\sup\limits_{r\in[t-\varepsilon,t]}\mathbf{Z}^{W,\mathrm{c}}_{\hat\beta,\hat h}(0, r)$ can be bounded by $\mathbf{Z}^{W,\mathrm{c}}_{\hat\beta,\hat h}(0, t)$ by paying a random factor which does give contributions to the free energy. For this purpose let us fix $r \in [t-\varepsilon, t]$ and decompose

$\mathbf{Z}^{W,c}_{\hat\beta,\hat h}(0,r)$ and $\mathbf{Z}^{W,c}_{\hat\beta,\hat h}(0,t)$ according to [24, (iv), theorem 2.4],

$$\frac{\mathbf{Z}^{W,c}_{\hat\beta,\hat h}(0,r)}{\mathbf{Z}^{W,c}_{\hat\beta,\hat h}(0,t)} = \frac{r^{1-\alpha}\displaystyle\int_0^{t-1} \mathrm{d}u\, C_\alpha \frac{\mathbf{Z}^{W,c}_{\hat\beta,\hat h}(0,u)}{u^{1-\alpha}}\left[\int_{t-1}^{r}\mathrm{d}v\,\frac{\mathbf{Z}^{W,c}_{\hat\beta,\hat h}(v,r)}{(v-u)^{1+\alpha}(r-v)^{1-\alpha}}\right]}{t^{1-\alpha}\displaystyle\int_0^{t-1} \mathrm{d}u\, C_\alpha \frac{\mathbf{Z}^{W,c}_{\hat\beta,\hat h}(0,u)}{u^{1-\alpha}}\left[\int_{t-1}^{t}\mathrm{d}v\,\frac{\mathbf{Z}^{W,c}_{\hat\beta,\hat h}(v,t)}{(v-u)^{1+\alpha}(t-v)^{1-\alpha}}\right]}. \tag{6.40}$$

In the internal integral we bound $\mathbf{Z}^{W,c}_{\hat\beta,\hat h}(v,r)$ and $\mathbf{Z}^{W,c}_{\hat\beta,\hat h}(v,t)$ from above and below respectively by k_r and c_t, cf. (6.21) and (6.18). Moreover we observe that

$$\frac{\displaystyle\int_{t-1}^{r}\frac{\mathrm{d}v}{(v-u)^{1+\alpha}(r-v)^{1-\alpha}}}{\displaystyle\int_{t-1}^{t}\frac{\mathrm{d}v}{(v-u)^{1+\alpha}(t-v)^{1-\alpha}}} \le \frac{\displaystyle\int_{t-1}^{r}\frac{\mathrm{d}v}{(v-u)^{1+\alpha}(r-v)^{1-\alpha}}}{\displaystyle\int_{t-1}^{r}\frac{\mathrm{d}v}{(v-u)^{1+\alpha}}}. \tag{6.41}$$

By Lemma 6.4 such ratio is bounded by a constant ξ depending on α and ε, uniformly on all $r \in [t-\varepsilon, t]$ and $u < t-1$. Summarizing we have obtained that $\mathbf{Z}^{W,c}_{\hat\beta,\hat h}(0,r) \le \xi\frac{k_r}{c_t}\mathbf{Z}^{W,c}_{\hat\beta,\hat h}(0,t)$. This provides an upper bound for (6.39), from which we get the upper bound for the continuum (free) partition function:

$$\mathbf{Z}^{W}_{\hat\beta,\hat h}(t) \le \xi\frac{\sup_{r\in[t-\varepsilon,t]} k_r}{c_t}\left(\frac{\alpha\, t^\alpha}{c_t\,\varepsilon^\alpha} + 1\right)\mathbf{Z}^{W,c}_{\hat\beta,\hat h}(0,t). \tag{6.42}$$

By using the properties of k_t and c_t, cf. Section 6.2, we have that $\frac{1}{t}\log\left[\xi\frac{\sup_{r\in[t-\varepsilon,t]}k_r}{c_t}\left(\frac{\alpha\,t^\alpha}{c_t\,\varepsilon^\alpha}+1\right)\right]$ converges to 0 as $t\to\infty$, a.s. and in L^1, see Remark 6.5. This concludes the proof.

Remark 6.5. To prove that $\frac{1}{t}\log\left[\xi\frac{\sup_{r\in[t-\varepsilon,t]}k_r}{c_t}\left(\frac{\alpha\,t^\alpha}{c_t\,\varepsilon^\alpha}+1\right)\right]$ converges to 0 as $t\to\infty$, a.s. and in L^1, we only need to show that $\frac{1}{t}\log\left(\sup_{r\in[t-\varepsilon,t]}k_r\right)\to 0$ as $t\to\infty$, a.s. and in L^1, because the convergence of the remaining part follows directly by Section 6.2. The convergence a.s. follows by the following (deterministic lemma):

Lemma 6.6. *Let f be a real function such that $\lim_{t\to\infty} f(t) = a^* \in \mathbb{R}$, then for any fixed $\varepsilon > 0$ the function $g(t) = \sup_{s\in[t-\varepsilon,t]} f(s)$ converges to the same limit a^* as $t\to\infty$.*

Proof. Since $a^*\in\mathbb{R}$, if t is large enough, we have that f is bounded and so g. For any fixed $\delta > 0$, there exists a point $t^* \in [t-\varepsilon, t]$ such that $f(t) \le g(t) \le f(t^*) + \delta$. It follows that $a^* \le \liminf_{t\to\infty} g(t) \le \limsup_{t\to\infty} g(t) \le a^* + \delta$. By the arbitrariness of δ follows the result. □

The convergence in L^1 follows by observing that $|\log x| \le x + 1/x$ and $k_r \ge 1$ because $\mathbf{Z}^{W,c}_{\hat\beta,\hat h}(r,r) = 1$, cf. (6.21). Therefore by definition of k_r, for any $\varepsilon \in (0, \frac{1}{4})$,

$$\mathbb{E}\left[\log\left(\sup_{r\in[t-\varepsilon,t]} k_r\right)\right] \le \mathbb{E}\left[\sup_{r\in[t-\varepsilon,t]} k_r\right] + 1 \le \mathbb{E}\left[\sup_{(v,r)\in[t-2,t]^2_\le} \mathbf{Z}^{W,c}_{\hat\beta,\hat h}(v,r)\right] + 1 = \mathbb{E}\left[\sup_{(v,r)\in[0,2]^2_\le} \mathbf{Z}^{W,c}_{\hat\beta,\hat h}(v,r)\right] + 1 < \infty$$

by Corollary 3.13.

□

Remark 6.7. Let us stress that (6.32) is the continuum limit of the analogous formula in the discrete case: let $t = 1$ for simplicity, then, starting from the definition (2.28) of $Z^{\omega}_{\beta_N,h_N}(N)$, we decompose according to the last visited point in $[0, N]$ by the renewal process

$$Z^{\omega}_{\beta_N,h_N}(N) = \sum_{x \in \frac{\mathbb{N}}{N} \cap [0,1]} Z^{\omega,\mathtt{c}}_{\beta_N,h_N}(0, Nx)\, u(Nx)\, \bar{K}(N - Nx)), \tag{6.43}$$

we observe that under a suitable coupling of the disorder, the l.h.s. of (6.43) converges uniformly to $\mathbf{Z}^{W}_{\hat{\beta},\hat{h}}(1)$. To control the r.h.s. we note that $u(Nx)\, \bar{K}(N - Nx)) \sim \frac{C_\alpha}{\alpha} x^{-(1-\alpha)}(1-x)^{-\alpha}$ as $N \to \infty$, therefore by using the uniform convergence of the partition function $Z^{\omega,\mathtt{c}}_{\beta_N,h_N}(0, Nx)$ to $\mathbf{Z}^{W,\mathtt{c}}_{\hat{\beta},\hat{h}}(0, x)$, a Riemann sum argument similar to [24, Section 2.3] provides the convergence to the r.h.s. of (6.32).

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Colophon

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Declaration

I hereby declare that this thesis is my original work and it has been written by me in its entirety. I have acknowledged all the sources of in formation which have been used in the thesis.

Lyon, 18 Septembre 2015

Niccolò Torri

Localization and universality phenomena for random polymers

Abstract:
A polymer is a long chain of repeated units (monomers) that are almost identical, but they can differ in their degree of affinity for certain solvents. Such property allows to have interactions between the polymer and the external environment. The environment has only a region that can interact with the polymer. This interaction can attract or repel the polymer, by changing its spatial configuration, giving rise to localization and concentration phenomena. It is then possible to observe the existence of a phase transition. Whenever such region is a point or a line (but also a plane or hyper-plane), then we talk about pinning model, which represents the main subject of this thesis. From a mathematical point of view, the pinning model describes the behavior of a Markov chain in interaction with a distinguished state. This interaction can attract or repel the Markov chain path with a force tuned by two parameters, h and β. If $\beta = 0$ we obtain the homogeneous pinning model, which is completely solvable. The disordered pinning model, which corresponds to $\beta > 0$, is most challenging and mathematically interesting. In this case the interaction depends on an external source of randomness, independent of the Markov chain, called disorder. The interaction is realized by perturbing the original Markov chain law via a Gibbs measure (which depends on the disorder, h and β), biasing the probability of a given path. Our main aim is to understand the structure of a typical Markov chain path under this new probability measure. The first research topic of this thesis (Chapter 4) is the pinning model in which the disorder is heavy-tailed and the return times of the Markov chain have a sub-exponential distribution. This work has interesting connections with the directed polymer in random environment with heavy tail. In our second result (Chapters 5-6) we consider a pinning model with a light-tailed disorder and the return times of the Markov chain with a polynomial tail distribution, with exponent tuned by $\alpha > 0$. It is possible to show that there exists a non-trivial interaction between the parameters h and β. Such interaction gives rise to a critical point, $h_c(\beta)$, depending only on the law of the disorder and of the Markov chain. Our goal is to understand the behavior of the critical point $h_c(\beta)$ in the weak disorder regime, namely for $\beta \to 0$. The answer depends on the value of α and in the literature there are precise results only for the case $\alpha < 1/2$ and $\alpha > 1$. We show that for $\alpha \in (1/2, 1)$ the behavior of the pinning model in the weak disorder limit is universal and the critical point, suitably rescaled, converges to the related quantity of a continuum model.

keywords: Pinning Model; Random Polymer; Directed Polymers; Weak Disorder; Disorder Relevance; Localization; Heavy Tails; Universality; Free Energy; Critical Point; Coarse-Graining

Phénomènes de localisation et d'universalité pour des polymères aléatoires

Résumé: D'un point de vue chimique et physique, un polymère est une chaîne d'unités répétées, appelées monomères, qui sont presque identiques, et chacune peut avoir un degré différent d'affinité avec certains solvants. Cette caractéristique permet d'avoir des interactions entre le polymère et le milieu dans lequel le polymère se trouve. Dans le milieu il y a une région interagissant, de manière attractive ou répulsive, avec le polymère. Cette interaction peut avoir un effet substantiel sur la structure du polymère, en donnant lieu à des phénomènes de localisation et de concentration et il est donc possible observer l'existence d'une transition de phase. Quand cette région est un point ou une ligne (ou alors un plan ou un hyper-plan) on parle du modèle d'accrochage de polymère qui représente l'objet d'étude principal de cette thèse. Mathématiquement le modèle d'accrochage de polymère décrit le comportement d'une chaîne de Markov en interaction avec un état donné. Cette interaction peut attirer ou repousser le chemin de la chaîne de Markov avec une force modulée par deux paramètres, h et β. Quand $\beta = 0$ on parle de modèle homogène, qui est complètement solvable. Le modèle désordonné, qui correspond à $\beta > 0$, est mathématiquement le plus intéressant. Dans ce cas l'interaction dépend d'une source d'aléa extérieur indépendant de la chaîne de Markov, appelée désordre. L'interaction est réalisée en modifiant la loi originelle de la chaîne de Markov par une mesure de Gibbs (dépendant du désordre, de h et de β), en changeant la probabilité d'une trajectoire donnée. La nouvelle probabilité obtenue définit le modèle d'accrochage de polymère. Le but principal est d'étudier et de comprendre la structure des trajectoires typiques de la chaîne de Markov sous cette nouvelle probabilité. Le premier sujet de recherche de cette thèse (Chapitre 4) concerne le modèle d'accrochage de polymère où le désordre est à queues lourdes et où le temps de retour de la chaîne de Markov suit une distribution sous-exponentielle. Ce travail a des connections intéressantes avec un autre modèle de polymère très répandu: le modèle de polymère dirigé en milieu aléatoire avec queues lourdes. Dans notre deuxième résultat (Chapitres 5-6) nous étudions le modèle d'accrochage de polymère avec un désordre à queues légères et le temps de retour de la chaîne de Markov avec une distribution à queues polynomiales avec exposant caractérisé par $\alpha > 0$. Sous ces hypothèses on peut démontrer qu'il existe une interaction non-triviale entre les paramètres h et β qui donne lieu à un point critique, $h_c(\beta)$, dépendant uniquement de la loi du désordre et de la chaîne de Markov. Nous cherchons à comprendre le comportement du point critique $h_c(\beta)$ dans la limite du désordre faible, i.e. quand $\beta \to 0$. La réponse dépend de la valeur de α et dans la littérature on a des résultats précises pour $\alpha < 1/2$ et $\alpha > 1$. Nous montrons que pour $\alpha \in (1/2, 1)$ le comportement du modèle d'accrochage de polymère dans la limite du désordre faible est universel et le point critique, opportunément changé d'échelle, converge vers la même quantité donnée par un modèle continu.

Mots clés: Modèle d'accrochage de Polymère; Polymères Aléatoires; Polymères Dirigés; Désordre Faible; Pertinence du désordre; Localisation; Queues Lourdes; Universalité; Énergie Libre; Point Critique; Coarse-Graining

Localization and universality phenomena for random polymers

keywords: Pinning Model; Random Polymer; Directed Polymers; Weak Disorder; Disorder Relevance; Localization; Heavy Tails; Universality; Free Energy; Critical Point; Coarse-Graining

Image en couverture : Coarse-Graining decomposition.

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INFOMATHS

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